JKLMR conjecture and Batyrev's polytopes.

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Preface

Using the Batyrev mirror construction we study the mirror version of the relation proposed by Jockers et all
between the exact partition functions of $N = (2, 2)$ gauged linear sigma-models on S^2 computed by Supersymmetric Localization and the Kähler potentials on the complex moduli spaces of the CY manifolds.

Introduction

As it was shown by Candelas et al in order to obtain the space-time supersymmetric effective theory in 4 dimensions it is necessary to compactify the superstring theory on Calabi-Yau manifolds.

The dynamics of the massless sector of this theory is governed by Kähler potentials of the moduli spaces of CY. Therefore for studing the low energy dynamics one has to compute the CY moduli geometry.

Taking into account this goal we study the mirror version of the recently discovered by Jockers et al relation (JKLMR conjecture) between the Kähler potentials on CY moduli space and the exact partition functions of the $N = (2, 2)$ gauged linear sigma-models (GLSM) on S^2 computed by Benini, Cremonesi and Doroud et al.

Main point

We will use the Batyrev mirror construction for finding the explicit correspondence between a GLSM with its gauge group and its set of the chiral superfields on the one hand side and the family of the Calabi-Yau manifolds X, defined as the hypersurfaces in the weighted projective spaces $\mathbb{P}^4_{k_1,k_2,k_3,k_4,k_5}$ on another hand side.

The key point of our approach is as follows. Let a CY hypersurface X be given by zeros of the superpotential

$$W_X(x_1,x_2,x_3,x_4,x_5|\psi_1,\ldots,\psi_h) = \sum_{a=1}^{h+5} C_a \prod_{i=1}^{5} x_i^{v_{ai}}, \; \sum_{i=1}^{5} k_i v_{ai} = \sum k_i.$$

Here h is equal to the Hodge number h_{21}. The coefficients C_a are some functions of the complex structure moduli $\psi_1,\ldots,\psi_h$.

Main point. Mirror symmetry and toric geometry

In Batyrev approach the set of the exponents v_{ai} corresponds to the lattice points of the *Reflexive polytope* Δ_X. They are coordinates of the 5-vectors $\vec{V}_a \in R^5$, that is $v_{ai} = (\vec{V}_a)_i$. These vectors $\vec{V}_a$ being 5-dimensional are subjects to the linear relations

$$\sum_{a=1}^{h+5} Q_{al} \vec{V}_a = 0, \quad l = 1, ..., h.$$

The Q_{al} is a set of integer numbers which corresponds to the relations space between the exponents of the monomials in W_X.

These vectors $\vec{V}_a$ are the "edges" of the *Fan* that defines the toric manifold $\mathbb{C}^N//(\mathbb{C}^*)^h$, where $N = h + 5$. The symbol $\mathbb{C}^N//(\mathbb{C}^*)^h$ denotes the quotient $(\mathbb{C} - Z)/(\mathbb{C}^*)^h$, where Z is a certain invariant subset.

This toric manifold can be defined by a set of the projective coordinates $y_1, ..., y_N \in \mathbb{C}^N$, which are subject to h identifications

$$(y_1, \ldots, y_N) \sim (\lambda^{Q_{1l}} y_1, \ldots, \lambda^{Q_{Nl}} y_N), \quad l = 1, \ldots, h.$$

Main point. Mirror symmetry and toric geometry

Then the Calabi-Yau manifold Y, which is the mirror of X, is realized as a surface in the toric manifold $\mathbb{C}^N//(\mathbb{C}^*)^h$ defined by the critical locus of a polynomial $W_Y(y_1, ..., y_N)$ which is weighted homogeneous with respect to all h dilatations

$$W_Y(\lambda^{Q_{1l}} y_1, \ldots, \lambda^{Q_{Nl}} y_N) = W_Y(y_1, \ldots, y_N), \quad l = 1, \ldots, h.$$

The natural coordinates $(z_1, \ldots, z_{N-h})$ on the toric manifold $\mathbb{C}^N//(\mathbb{C}^*)^h$, invariant with respect to the group action, can be taken the monomials $z_j = \prod_{a=1}^N y_a^{v_{aj}}$. Thus W_Y can be written as their linear combination

$$W_Y(y_1, \ldots, y_N) = \sum_{j=1}^{5} \tilde{C}_j \prod_{a=1}^{N} y_a^{v_{aj}}$$

The manifold Y given by this equation corresponds to the reflexive polytope Δ_Y that is polar dual to the polytope Δ_X.

Main point. GLSM

Calabi-Yau manifolds of such type arise as supersymmetric vacua manifold in 2d $N = (2,2)$ gauged linear sigma-models (GLSM).

The weights Q_{al} are just the charges of N chiral superfields $(\Phi_1, \ldots, \Phi_N)$ under the gauge groups $U(1)_l$.

Thus, knowing the family of CY manifolds X defined as the hypersurfaces a weighted projective by a polynomial W_X, we can find the corresponding $N = (2,2)$ gauged linear sigma-model.

We find the charges $\{Q_{al}\}$ by solving the equation

$$\sum_{a=1}^{h+5} Q_{al} \vec{V}_a = 0, \quad l = 1, ..., h.$$

Vacua in GLSM and hypersurfaces in Toric manifolds

Here we shortly recollect the connection between the supersymmetric vacua in GLSM and the hypersurfaces in toric manifolds.

Consider the $U(1)^h = \prod_{l=1}^{h} U(1)_l$ $N = (2,2)$ gauge model h vector superfields and N chiral matter fields $(\Phi_1, \ldots \Phi_N)$ whose charges under $U(1)_l$ are denoted by $(Q_{1l}, \ldots, Q_{Nl})$.

Lagrangian of the model also depends on coupling constants $(e_1, \ldots, e_h)$, the real Fayet–Iliopoulos (FI) parameters r_l, $l = 1, \ldots, k$, the theta angles and the superpotential W_Y.

Vacua in GLSM and hypersurfaces in Toric manifolds

The potential term for the scalar fields in this Lagrangian is

$$U(\phi) = \sum_{l=1}^{h} \frac{e_l^2}{2} \left(\sum_{a=1}^{N} Q_{al} |\phi_a|^2 - r_l \right)^2 + \frac{1}{4} \sum_{a=1}^{N} \left| \frac{\partial W_Y}{\partial \phi_a} \right|^2 ,$$

The supersymmetric ground states of the theory are parametrized by the minima of the potential modulo gauge equivalences.

$$Y_{\mathbf{r}} = \left\{ (\phi_1, \ldots \phi_N) \middle| \sum_{a=1}^{N} Q_{al} |\phi_a|^2 = r_l,\ l = 1, \ldots h,\ \frac{\partial W_Y}{\partial \phi_a} = 0 \right\} / U(1)^h.$$

If we send all coupling constants e_l to infinity the model reduces to the $N = (2,2)$ non-linear sigma model with the target space $Y_{\mathbf{r}}$.

The toric manifolds $\mathbb{C}^N // (\mathbb{C}^*)^h$ and $Y_{\mathbf{r}}$ are equivalent.

This establishes a one-to-one correspondence between the hypersurfaces Y in toric manifolds and the gauged linear sigma-models (GLSM).

It was shown by Benini et al and Doroud et al that GLSM can be put on the 2-sphere while preserving the $N=(2,2)$ supersymmetry, which allows one to compute the partition function of this theory exactly,
using the Supersymmetric Localization

$$Z_Y = \sum_{m_l \in \mathbb{Z}} \prod_{l=1}^{h} e^{-i\theta_l m_l} \times$$

$$\times \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_k} \prod_{l=1}^{h} \frac{d\tau_l}{(2\pi i)} e^{4\pi r_l \tau_l} \prod_{a=1}^{N} \frac{\Gamma\Big(q_a/2 + \sum_{l=1}^{k} Q_{al}(\tau_l - \frac{m_l}{2})\Big)}{\Gamma\Big(1 - q_a/2 - \sum_{l=1}^{k} Q_{al}(\tau_l + \frac{m_l}{2})\Big)},$$

where the contours $\mathcal{C}_l$ go along the imaginary axis. The partition function does not depend either on the coupling constants $(e_1, \ldots, e_h)$ or on the details of the polynomial W_Y.

JKLMR conjecture

Thus effectively Z_Y picks up only the contribution of the massless fields and gives the exact partition function of the non-linear sigma-model.
It was conjectured by Jockers et all that the exact expression for the partition function coincides with the Kähler potential $K_K^Y = -\log Z_Y$ on the quantum Kähler moduli space for Calabi-Yau manifold Y.

This conjecture was verified for a few examples. The problem with this check is the lack of simple definition of the function K_K^Y.

Here we will adopt a different point of view based on mirror symmetry and Batyrev approach to it.

Mirror version of JKLMR conjecture

The mirror symmetry implies an equality

$$K_K^Y = K_C^X, \qquad e^{-K_C^X} = -i \int_X \Omega \wedge \bar{\Omega},$$

where K_C^X is the Kähler potential for complex structure moduli of the mirror manifold X.

The last potential has a transparent definition in terms of integral of the uniquely defined holomorphic form Ω on X.

The manifold X is canonically related to the manifold Y realized as a hypersurface in toric manifold by the mirror map.

So the conjecture of Jockers et al can be reformulated as

$$\int_X \Omega \wedge \bar{\Omega} = iZ_Y.$$

We will call this the mirror version of JKLMR conjecture.

Batyrev construction

Consider the example of Quintic threefold.
In this case the CY X is defined as the hypersurface in the projective space $\mathbb{P}^4$, that is a set of five complex coordinates $(x_1, \ldots, x_5)$ identified by

$$(x_1, x_2, x_3, x_4, x_5) \sim (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5).$$

The hypersurface $X \in \mathbb{P}^4$ is given by the zeroes of the superpotential

$$W_X(x_1, x_2, x_3, x_4, x_5 | \psi_1, \ldots, \psi_h) = \sum_{i=1}^{5} x_i^5 + \sum_{l=1}^{h} \psi_l e_l(x),$$

$$e_l(x) = \prod_{i=1}^{5} x_i^{s_{li}}, \quad \text{with} \quad 0 \leq s_{li} \leq 3, \quad h = 101 \quad \text{and} \quad \sum_{i=1}^{5} s_{li} = 5.$$

The parameters ψ_l represent deformations of the complex structure of the manifold X.

Batyrev construction

The manifold Y, the mirror to X, can be realized as a hypersurface in the toric manifold defined as the quotient $\mathbb{C}^{h+5}//(\mathbb{C}^*)^h$.
The action of the group $(\mathbb{C}^*)^h$ depend of by the matrix Q_{al}, as was explained above.
Batyrev's construction tells us how to find this matrix.
The vectors V_a whose components are the exponents v_{ai} in W_X equal

$$v_{ai} = \begin{cases} 5\delta_{a,i}, & 1 \le a \le 5, \\ s_{a-5,i}, & 6 \le a \le h+5. \end{cases}$$

constitute the Fan of dual manifold Y.

As it was explained above one can find the charges Q_{al} of the chiral superfields in the corresponding GLSM by finding the linear dependence between these vectors

$$\sum_{a=1}^{h+5} Q_{al} \cdot V_a = 0.$$

The GLSM

The convenient choice of solutions for this equation is then

$$Q_{al} = \begin{cases} s_{la}, & 1 \leq a \leq 5, \\ -5\delta_{a-5,l}, & 6 \leq a \leq 106. \end{cases}$$

We can see that $\sum_{a=1}^{106} Q_{al} = 0$.

Therefore, as explained by Witten the FI parameters and the theta parameters do not run with RG flow and the $U(1)$ axial symmetry is unbroken. The FI and theta parameters r_l and θ_l remain arbitrary parameters also in the quantum theory.

The superpotential W_Y can be written in terms of the invariant coordinates as it is presented above.

It is convenient to introduce the new denotations for the 106 chiral matter fields $(\Phi_1, \ldots \Phi_{106})$ and their scalar components, whose charges under $U(1)_l$ we found, as follows

$$\Phi_a = \begin{cases} S_a, & 1 \leq a \leq 5, \\ P_{a-5}, & 6 \leq a \leq 106. \end{cases}$$

The GLSM

The field P_1 corresponds to v_{6i}, whose components $v_{6i} = 1$ will play a somehow distinguished role.The superpotential W_Y in the considered case can be written as

$$W_Y = P_1 \cdot G(S_1, \dots, S_5; P_2, \dots, P_{101}).$$

The potential term for the scalar fields in the Lagrangian, which was given above, will then take the form

$$\begin{aligned} U(\phi) = \sum_{l=1}^{101} \frac{e_l^2}{2} \left(\sum_{a=1}^{5} s_{al} |S_a|^2 - 5|P_l|^2 - r_l \right)^2 + \\ + \frac{1}{4} |G(S_1, \dots, S_5; P_2, \dots, P_{101})|^2 + \\ + \frac{1}{4} |P_1|^2 \sum_{a=1}^{5} \left| \frac{\partial G}{\partial S_a} \right|^2 + \frac{1}{4} |P_1|^2 \sum_{l=2}^{101} \left| \frac{\partial G}{\partial P_l} \right|^2 . \end{aligned}$$

The GLSM

Depending on the values of FI parameters in the theory the different phases are expected to occur.

In a suitable region of r_l the vacuum manifold is a set of (S_i, P_l) obeying the equations

$$\sum_{a=1}^{5} s_{al}|S_a|^2 - 5|P_l|^2 - r_l = 0,\ G(S_1, \ldots, S_5; P_2, \ldots, P_{101}) = 0,\ P_1 = 0$$

divided by the gauge group action.

This is the explicit definition of the hypersurface Y in the toric space Y_r, which is the vacuum manifold of the model.

Localization formula for the quintic threefold

In the case considered the exact partition function Z_Y can be written in the form

$$Z_Y = \sum_{m_l} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_{101}} \prod_{l=1}^{101} \frac{d\tau_l}{(2\pi i)} \left(z_l^{-\tau_l + \frac{m_l}{2}} \bar{z}_l^{-\tau_l - \frac{m_l}{2}} \right) \times$$

$$\frac{\Gamma\left(1 - 5(\tau_1 - \frac{m_1}{2})\right)}{\Gamma\left(5(\tau_1 + \frac{m_1}{2})\right)} \prod_{a=1}^{5} \frac{\Gamma\left(\sum_l s_{al}(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 - \sum_l s_{al}(\tau_l + \frac{m_l}{2})\right)} \prod_{l=2}^{101} \frac{\Gamma\left(-5(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 + 5(\tau_l + \frac{m_l}{2})\right)},$$

where

$$z_l = e^{-(2\pi r_l + i\theta_l)}.$$

The contours $\mathcal{C}$ go slightly to the left of the imaginary axes: $\tau_l = -\epsilon + it_l$. We consider the expansion of this integral for large values of $|z_l|$, that is for $r_l < 0$.

Check of JKLMR conjecture

For $r_l < 0$ each contour $\mathcal{C}_{\updownarrow}$ can be closed to the right half-plane picking up the poles at

$$5\left(\tau_l - \frac{m_l}{2}\right) = p_l;\ \ p_1 = 1, 2, \ldots,\ \ p_l = 0, 1 : \quad p_l + 5m_l > 0.$$

It is convenient to introduce $\bar{p}_l = p_l + 5m_l$ which varies from 0 to ∞. Then the partition function can be rewritten as

$$Z = \pi^{-5} \sum_{\bar{p}_l \geq 0} \sum_{p_l \geq 0} \prod_l \frac{(-1)^{p_l}}{p_l! \bar{p}_l!} z_l^{-\frac{p_l}{5}} \bar{z}_l^{-\frac{\bar{p}_l}{5}} \times$$
$$\times \prod_{i=1}^{5} \Gamma\left(\frac{1}{5} \sum_{l=1}^{h} s_{li} p_l\right) \Gamma\left(\frac{1}{5} \sum_{l=1}^{h} s_{li} \bar{p}_l\right) \sin\left(\frac{\pi}{5} \sum_{l=1}^{h} s_{li} \bar{p}_l\right).$$

Where the sum over $\bar{p}_l$ goes over the set which is restricted by the condition $\sum s_{li} \bar{p}_l = \sum s_{li} p_l \mod 5$ and the terms such that $\sum_{l=1}^{h} s_{il} \bar{p}_l = 0 \mod 5$ are removed.

Check of JKLMR conjecture

It means that the sum effectively goes over

$$S_{\boldsymbol{\mu},\mathbf{n}} = \left\{ p_l : \sum_{l=1}^{h} s_{li} p_l = \mu_i + 5n_i,\ 1 \le \mu_i \le 4\ , 0 \le n_i \right\}.$$

Obviously, if $\{p_l\}$ belongs to $S_{\boldsymbol{\mu}}$ so does the set $\{\bar{p}_l\}$. Using,

$$\prod_{i=1}^{5} \sin\left(\frac{\pi}{5}\sum_{l=1}^{h} s_{li}\bar{p}_l\right) = (-1)^{|\boldsymbol{\mu}|}\prod_{i=1}^{5}\sin\left(\frac{\pi\mu_i}{5}\right)\prod_{l=1}^{h}(-1)^{\bar{p}_l},$$

we find that

$$Z = \sum_{\boldsymbol{\mu}} (-1)^{|\boldsymbol{\mu}|} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{\mu_i}{5}\right)}{\Gamma\left(1-\frac{\mu_i}{5}\right)} |\sigma_{\boldsymbol{\mu}}(\mathbf{z})|^2,$$

$$\sigma_{\boldsymbol{\mu}}(\mathbf{z}) = \sum_{n_i \ge 0} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{\mu_i}{5}+n_i\right)}{\Gamma\left(\frac{\mu_i}{5}\right)} \sum_{\mathbf{p}\in S_{\boldsymbol{\mu},\mathbf{n}}} \prod_{l=1}^{101} \frac{(-1)^{p_l} z_l^{-\frac{p_l}{5}}}{p_l!}.$$

We find that Z exactly coincides with the expression for $\int_X \Omega \wedge \bar{\Omega}$ obtained by us earlier, if we identify the parameters as $z_l = \psi_l^{-5}$.

Conclusion

Thus, we have shown how, starting from the superpotential $W_X(x)$ which defines a CY manifold X, to find $N = (2,2)$ gauged linear sigma model whose vacuum manifold Y is the mirror to X. We have also verified here the conjecture by Jockers et al for the quintic threefold.

Actually, in the region $r_l < 0$, where the formula for the metric on the Complex moduli space of the CY manifold X was obtained, the theory is expected to describe the Landau-Ginzburg phase.

On the other hand, in order to get a metric on the Kähler moduli space of the manifold Y which is mirror to the X one need to perform an analytic continuation to other region of r_l'.

We note that the integral formula for the GLSM partition function can be used as a tool for this analytic continuation.