Notes on quantum fields in Static patch of de Sitter space

E.T.Akhmedov, K.V.Bazarov(speaker), D.V.Diakonov, U.Moschella

st. Bolshaya Cheremushkinskaya, 25, Institute of Theoretical and Experimental Physics, Moscow, 117218, Russia

Institutskiy per., 9, Moscow Institute of Physics and Technology, Dolgoprudny, 141700, Russia

Bazarov.KV@phystech.edu

Abstract

We show the explicit mode expansion of tree-level propagators in Static (or Compact) Patch of de Sitter space. We construct propagator for thermal state corresponding to arbitrary temperature T. We show that the propagator that respects the de Sitter isometry corresponds to the thermal state with $T=(2\pi)^{-1}$ in the units of de Sitter curvature. Which confirms the old and well known result, making it a bit more explicit. Propagators with $T\neq(2\pi)^{-1}$ do not respect the isometry. Moreover, we show that propagators with $T\neq(2\pi)^{-1}$ have extra singularities on the boundary of the Static Patch, as opposed to the case of $T=(2\pi)^{-1}$. We discuss physical meaning of these observations. We also discuss loop corrections to the propagators in the Static patch and their physical meaning both for $T=(2\pi)^{-1}$ and $T\neq(2\pi)^{-1}$.

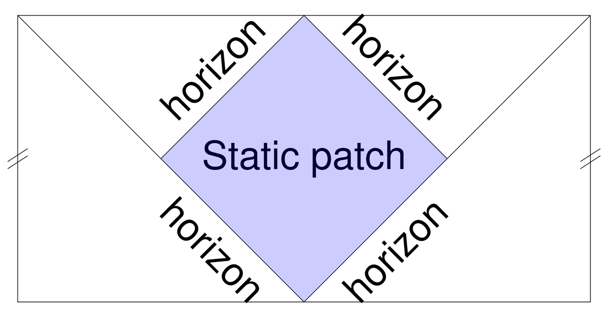
1. Geometry

The two dimensional de Sitter:

$$X_\mu X^\mu = X_0^2 - X_1^2 - X_2^2 = -1, \qquad ds_3^2 = dX_\mu dX^\mu$$

A suitable coordinate system for the static patch is:

$$\begin{cases} X^0 = \sinh t\,(\cosh x)^{-1} \\ X^1 = \tanh x \\ X^2 = \cosh t\,(\cosh x)^{-1} \end{cases}, \qquad t\in(-\infty,\infty),\ x\in(-\infty,\infty). \quad (1)$$



$$ds^2 = \frac{dt^2 - dx^2}{\cosh x^2}; \quad (2)$$

$$Z = Z_{12} = X_1^\mu X_{2\mu} = \frac{\cosh(t_1-t_2) + \sinh x_1 \sinh x_2}{\cosh x_1 \cosh x_2}. \quad (3)$$

2. Canonical Quantization

In this section we study the canonical quantization of the Klein-Gordon field

$$\left[\partial_t^2 - \partial_x^2 + \frac{m^2}{\cosh^2 x}\right]\varphi(t,x) = 0. \qquad \mu^2 = m^2 - \frac{1}{4}. \quad (4)$$

The mode expansion of the field operator $\hat\phi(t,x)$ can be represent as follows:

$$\hat\phi(t,x) = \int_0^\infty \frac{d\omega}{2\pi}\left[e^{-i\omega t}\Big(\psi_\omega(x)a_\omega + \psi_\omega(-x)b_\omega\Big) + e^{i\omega t}\Big(\psi^*_\omega(x)a^\dagger_\omega + \psi^*_\omega(-x)b^\dagger_\omega\Big)\right], \quad (5)$$

where $\psi_\omega(x)$ is:

$$\psi_\omega(x) = \sqrt{\sinh(\pi\omega)}\,\Gamma\Big(\frac{1}{2}+i\mu-i\omega\Big)\Gamma\Big(\frac{1}{2}+i\mu+i\omega\Big)\mathrm{P}^{i\omega}_{-\frac{1}{2}+i\mu}(\tanh x).$$

It is straightforward to show that:

$$\hat H_0 =: \int_{-\infty}^{+\infty} dx\sqrt{g}T^0_0 := \int_0^{+\infty} d\omega\ \omega\left(\hat a^\dagger_\omega \hat a_\omega + \hat b^\dagger_\omega \hat b_\omega\right). \quad (6)$$

3. Arbitrary temperature propagator

By definition the quantum average over the thermal state is defined as:

$$\left\langle \hat O\right\rangle = \frac{\mathrm{Tr}\,\hat\rho\hat O}{\mathrm{Tr}\,\hat\rho}, \qquad \hat\rho \equiv e^{-\beta\hat H}. \quad (7)$$

Wightman functions is as follows:

$$W_\beta(x_1,x_2|t=t_2-t_1) = \int_{-\infty}^{\infty}\frac{d\omega}{4}\frac{1}{e^{\beta\omega}-1}\frac{\sinh\pi\omega}{\cosh\pi(\omega-\mu)\cosh\pi(\omega+\mu)}e^{i\omega t}\times$$
$$\times\left(\mathrm{P}^{i\omega}_{-\frac{1}{2}+i\mu}(\tanh x_1)\mathrm{P}^{-i\omega}_{-\frac{1}{2}+i\mu}(\tanh x_2) + \mathrm{P}^{i\omega}_{-\frac{1}{2}+i\mu}(-\tanh x_1)\mathrm{P}^{-i\omega}_{-\frac{1}{2}+i\mu}(-\tanh x_2)\right). \quad (8)$$

$$W_{2\pi}(x_1,x_2|t=t_2-t_1) = W_{2\pi}\big(Z_{12}\big) = \frac{1}{4\cosh\mu\pi}P_{-\frac{1}{2}+i\mu}(-Z_{12}). \quad (9)$$

4. Behavior of Wightman function for large time separation

The integrand in (8) has the following poles:

$$w = i\frac{2\pi}{\beta}n, \qquad n\in\mathbb{Z}, \qquad \text{coming from} \quad \frac{1}{e^{\beta\omega}-1}, \quad (10)$$

and

$$w = \pm\mu - \frac{i}{2} + in, \qquad n\in\mathbb{Z} \qquad \text{coming from} \quad \frac{1}{\cosh\pi(\omega-\mu)\cosh\pi(\omega+\mu)}. \quad (11)$$

$$W_\beta(x=y=0|t\to\infty) \approx \begin{cases} C_+ e^{-t(\frac{1}{2}-i\mu)} + C_- e^{-t(\frac{1}{2}+i\mu)} & \text{,if } \beta<4\pi \\ C_\beta e^{-t\frac{2\pi}{\beta}} & \text{,if } \beta>4\pi \end{cases}. \quad (12)$$

5. One loop corrections

Consider the following self-interaction: $L_{\text{int}} = \lambda\varphi(t,x)^3$. At large distances one loop correction is as follows:

$$\hat G(t) = \lambda^2 t\Big(\hat C_1 e^{-t(\frac{1}{2}-i\mu)} + \hat C_2 e^{-t(\frac{1}{2}+i\mu)}\Big), \qquad t\to\infty. \quad (13)$$

6. Behavior of Wightman function, when their arguments are light-like separated

It is natural to expect that when $t=\pm(x_2-x_1)$ any propagator behaves as in the Minkowski space. Then,

$$W_{2\pi}(Z\approx 1) = \frac{|\Gamma(\frac{1}{2}+i\mu)|^2}{4\pi}P_{-\frac{1}{2}+i\mu}(-Z\approx -1) \approx \frac{1}{4\pi}\log(1-Z) \approx \frac{1}{4\pi}\log\left[t^2-(x_2-x_1)^2\right]. \quad (14)$$

One can see the same picture from the mode expansion of the Wightman function for arbitrary β. At light-like separation (small distances) large ω dominate in the integral in (8). Then a straightforward calculation gives:

$$W_\beta(x_1,x_2|t) \approx -\int_{-\infty}^{0}\frac{d\omega}{\pi}\frac{e^{i\omega t}}{2\omega}\Big(e^{i\omega(x_2-x_1)} + e^{-i\omega(x_2-x_1)}\Big) = \frac{1}{4\pi}\log\left[t^2-(x_2-x_1)^2\right].$$

7. Singular behaviour of Wightman functions for arbitrary temperature

Unlike the case $\beta=2\pi$ for arbitrary β the propagator $W_\beta(x_1,x_2|t)$ is not a function of the invariant (3) anymore $W_\beta(x_1,x_2|t)\neq W(Z_{12})$. But for the case when $\beta=\frac{2\pi}{n}$ it acquires quite a simple form. Let us consider the following chain of relation:

$$\frac{e^{i\omega t}}{e^{2\pi\omega/n}-1} = \frac{e^{i\omega t}}{e^{2\pi\omega}-1}\frac{e^{2\pi\omega}-1}{e^{2\pi\omega/n}-1} =$$
$$= \frac{e^{i\omega t}}{e^{2\pi\omega}-1}\left[e^{2\pi\omega\frac{n-1}{n}} + e^{2\pi\omega\frac{n-2}{n}} + \ldots + e^{2\pi\omega\frac{1}{n}} + 1\right] = \sum_{k=1}^{n}\frac{e^{i\omega\left(t+i2\pi\frac{k}{n}\right)}}{e^{2\pi\omega}-1}. \quad (15)$$

Using it we obtain that:

$$W_{\beta=\frac{2\pi}{n}}\Big(x_1,x_2|t\Big) = \sum_{k=1}^{n} W_{2\pi}\Big(x_1,x_2|t+i2\pi\frac{k}{n}\Big). \quad (16)$$

For example for the simplest case $\beta=\pi$ we have only two terms:

$$W_{\beta=\pi}\Big(x_1,x_2|t\Big) = \frac{|\Gamma(\frac{1}{2}+i\mu)|^2}{4\pi}\left[P_{-\frac{1}{2}+i\mu}\left(-\frac{\cosh(t)+\sinh(x_1)\sinh(x_2)}{\cosh(x_1)\cosh(x_2)}\right)+\right.$$
$$\left.+P_{-\frac{1}{2}+i\mu}\left(-\frac{-\cosh(t)+\sinh(x_1)\sinh(x_2)}{\cosh(x_1)\cosh(x_2)}\right)\right]. \quad (17)$$

Tthe first term in (17) coincides with the dS invariant Wightman funcion. It has singularity only on the light cone. Let us remind that Legendre function has a singularity when its argument is equal to minus one. Consider the following situation:

$$x_1=\lambda, \qquad x_2=\lambda+\Delta x, \qquad t_1=\lambda+c_1, \qquad t_2=\lambda+c_2. \quad (18)$$

The horizon corresponds to the limit:

$$\lambda\to\infty.$$

Then the arguments of Legendre functions in (17) are as follows:

$$\frac{\pm\cos(c_2-c_1)+\sinh(\lambda)\sinh(\lambda+\Delta x)}{\cosh(\lambda)\cosh(\lambda+\Delta x)} = 1-4e^{-\lambda}\approx 1, \quad \text{as} \quad \lambda\to\infty.$$

It means that both terms in (17) become singular on the horizon. Thus, we have an extra singularity on the horizon. One can see the same fact from the mode expansion of $W_\beta(x_1,x_2|t)$:

$$W_\beta(x_1,x_2|t) \approx \int d\omega\frac{e^{i\omega(c_2-c_1)}}{e^{\beta(\omega+i\epsilon)}-1}\Big(C_+(\omega)e^{2i\omega\lambda} + C_-(\omega)e^{-2i\omega\lambda}\Big).$$

The leading contribution to the integral comes from the pole at $\omega=-i0$:

$$W_\beta(\lambda\to\infty) \approx -\frac{1}{2}\int d\omega\frac{1}{\left(e^{\beta(\omega+i\epsilon)}-1\right)\sinh\pi(\omega+i\epsilon)}e^{-2i\omega\lambda}.$$

The answer is as follows:

$$W_\beta(\lambda\to\infty) \approx \frac{2\pi}{\beta}\frac{\lambda}{\pi}.$$

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