Production Gravitational Waves

concentrate on the spatial components

$$\bar{h}_{ik}(t,\vec{x}) \approx \frac{4G_N}{r}\int d^3y\, T^{ret}_{ik}(t,\vec{y})$$

$$\bar{h}_{ik}(t,\vec{x}) \approx \frac{2G_N}{r}\ddot{Q}^{ret}_{ik} \qquad\qquad Q^{ret}_{ik}(t) = \int d^3x\ \rho^{ret} x_i x_k$$

In case when $\rho(t) \sim \mathrm{e}^{-i\Omega t}$ $\Rightarrow$ $\bar{h}_{ik}(t,r) \approx -2G_N\Omega^2 Q^{ret}_{ik}\,\dfrac{\mathrm{e}^{-i\Omega(t-r)}}{r}$

an outgoing spherical wave.

$$\boxed{\frac{dE}{dt} = -\frac{G_N}{5}(\dddot{\mathcal{Q}}^{ret})_{ik}(\dddot{\mathcal{Q}}^{ret})^{ik}}$$

$$\begin{aligned}\mathcal{Q}^{ret}_{ik} &= \int d^3x\ \rho^{ret}(x_i x_k - \tfrac{1}{3}\delta_{ik}r^2)\\ &= Q^{ret}_{ik} - \tfrac{1}{3}\delta_{ik}(Q^{ret})^j_{\ j}\ .\end{aligned}$$

Light geodesics in the Schwarzschild metric

Tangent vector to geodesic p^μ.

$$p^\mu \xi_\mu = C_K = const.$$

$$\begin{aligned}
\frac{d}{d\lambda}(p^\mu \xi_\mu) = \frac{D}{d\lambda}(p^\mu \xi_\mu) &= \frac{D^2 x^\mu}{d\lambda^2} \cdot \xi_\mu + p^\mu \frac{D\xi^\mu}{d\lambda} = p^\mu p^\nu \nabla_\nu \xi^\mu \\
&= \frac{1}{2} p^\nu p^\mu (\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu) = 0
\end{aligned}$$

$$p^\nu \nabla_\nu (p^\mu \xi_\mu) = p^\nu \nabla_\nu p^\mu \cdot \xi_\mu + p^\nu p^\mu \nabla_\nu \xi_\mu = 0$$

$$\begin{aligned}
\frac{d}{d\lambda}(p^\mu \xi_\mu) &= \frac{d}{d\lambda} p^\nu \cdot \xi_\mu + p^\nu \frac{d}{d\lambda} \xi_\mu = \frac{d^2 x^\mu}{d\lambda^2} \xi_\mu + p^\mu p^\alpha \partial_\alpha \xi_\mu \\
&= -\Gamma^\mu_{\alpha\beta} \xi^\mu p^\alpha p^\beta + p^\mu p^\alpha \partial_\alpha \xi_\mu = p^\nu p^\mu \nabla_\nu \xi_\mu = 0.
\end{aligned}$$

Light geodesics in the Schwarzschild metric

Motion in the equatorial plane, so $\theta = \pi/2$. Killing vectors:

$$\xi = \partial_t, \quad \xi^\mu = (1,0,0,0) \quad \rightarrow \quad \xi_\mu = \left(-\left(1 - \frac{2M}{r}\right), 0, 0, 0\right)$$

$$\eta = \partial_\phi, \quad \eta^\mu = (0,0,0,1) \quad \rightarrow \quad \xi_\mu = \left(0,0,0,r^2\right)$$

Light geodesics: $p^\mu p_\mu = 0$

$$-(1-2M/r)\dot{t}^2 + (1-2M/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = 0$$

Conservation of energy

$$E = -\xi_\alpha p^\alpha = (1-2M/r)\dot{t}$$

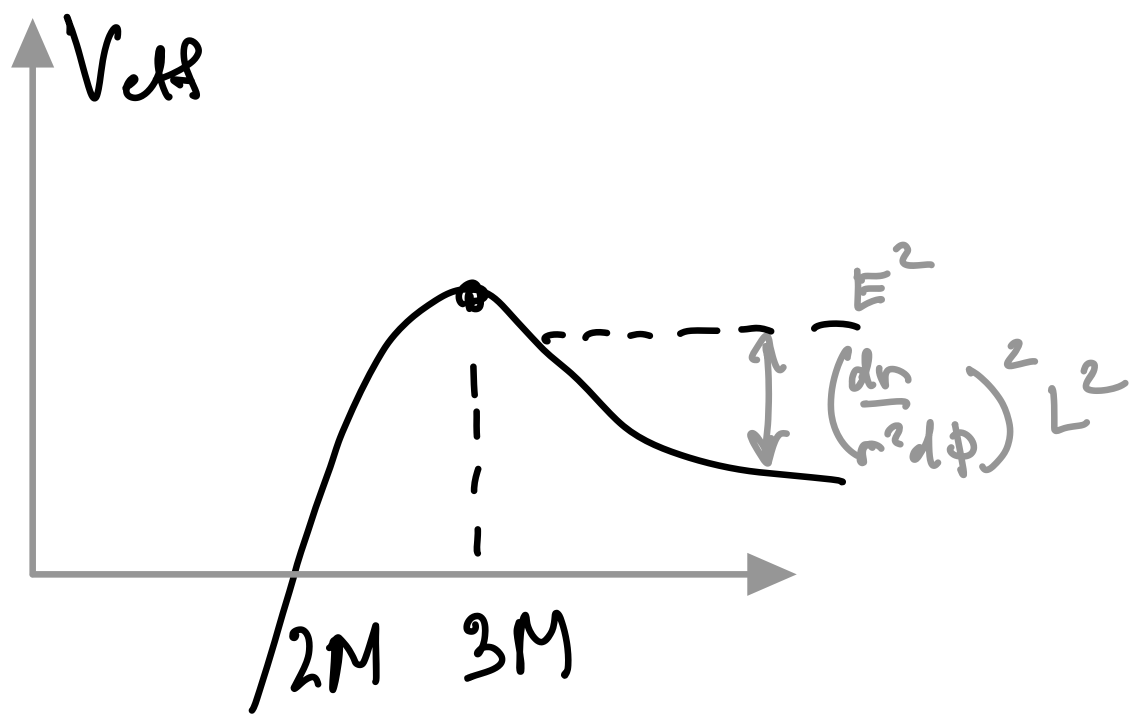
Angular momentum concervation

$$L = \eta_\alpha p^\alpha = r^2\dot{\phi}^2$$

Light geodesics in the Schwarzschild metric

Combining 3 equations:

$$\left(\frac{dr}{r^2 d\phi}\right)^2 L^2 = E^2 - \underbrace{\frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}_{V_{eff}}$$

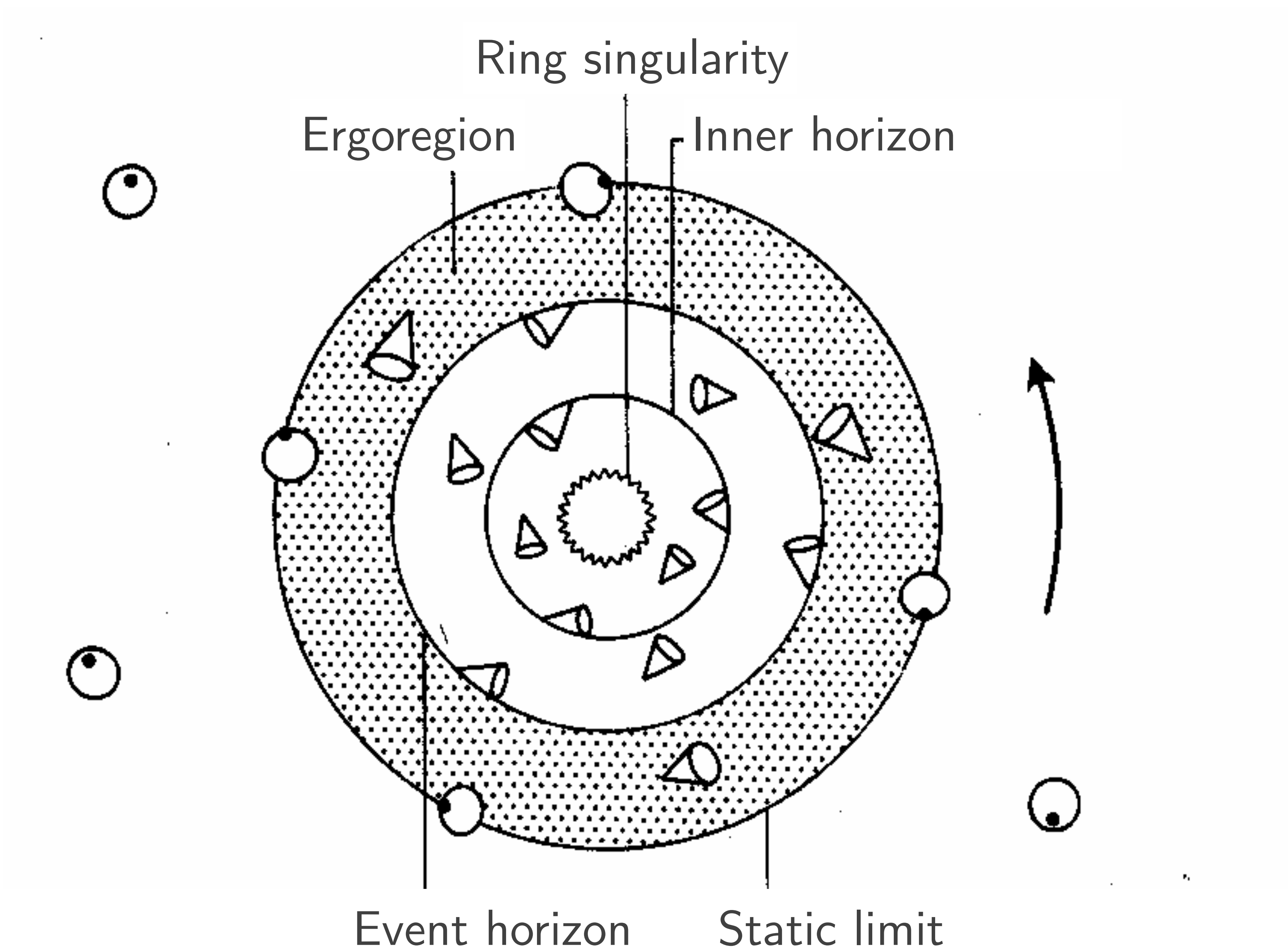


Photons accumulate around $r = 3M$

Bright ring around a black hole

M87*

Penrose process



Static observers do not exist inside the ergosphere.

The Killing vector $\partial/\partial t$ becomes spacelike (it is outside the light cone).

Penrose process

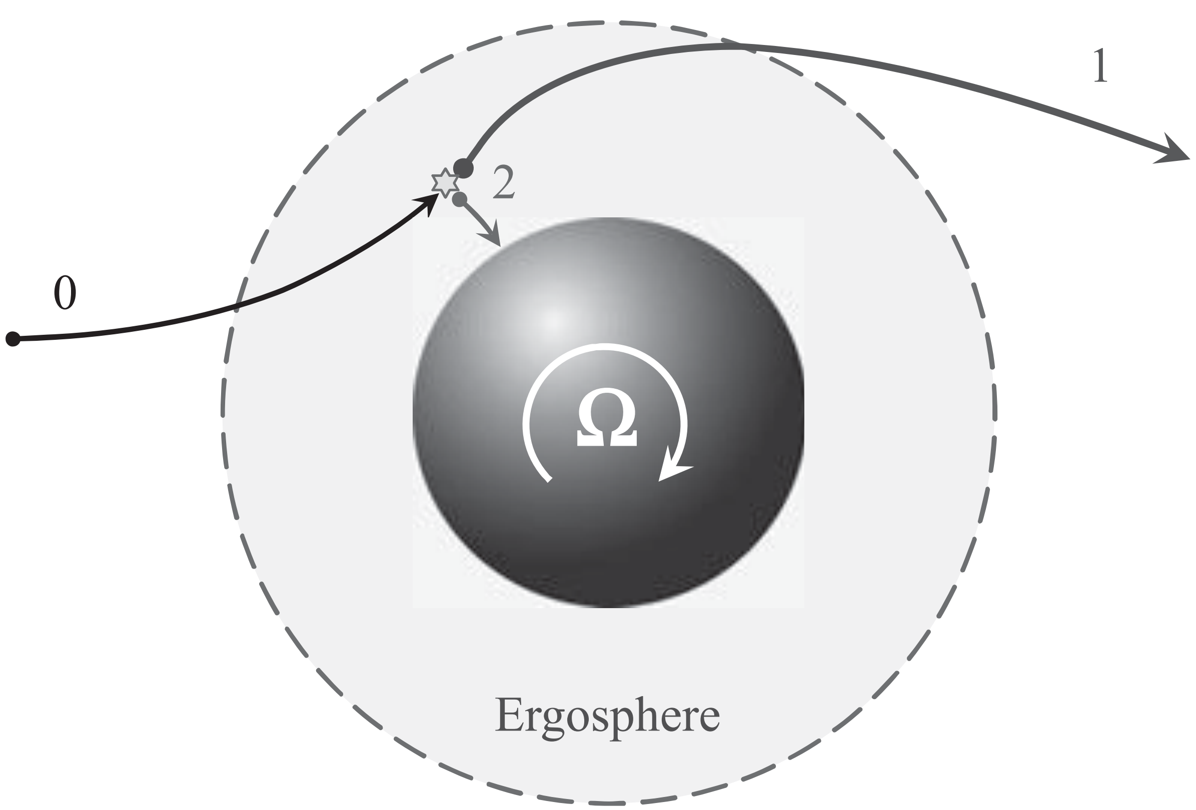
Static observers do not exist inside the ergosphere.

The Killing vector $\partial/\partial t$ becomes spacelike (it is outside the light cone).

The energy $E = -p^\mu \xi_\mu > 0$ for timelike future directed $\partial/\partial t$ and p^μ.

For the spacelike Killing vector E may be both positive or negative.

Negative energy particles cannot leave the ergosphere



A particle 0 with energy E_0 enters the ergosphere and decays there into two particles, 1 and 2.

The energy of particle 2 is negative (fall into the black hole)

In this process the energy is extracted.

$$\Delta E = E_1 - E_0 > 0$$

fig from the book by Frolov&Zelnikov

Gravitational waves and black holes

Perturbation of black holes

Test scalar field in the Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Equation of motion for a massless scalar field Φ

$$\Box\Phi \equiv (-g)^{-1/2}\partial_\mu \left[(-g)^{1/2} g^{\mu\nu}\partial_\nu\Phi\right] = 0$$

The metric is spherically symmetric, we introduce the mode decomposition

$$\Phi_{\ell m} = \frac{u_\ell(r,t)}{r} Y_{\ell m}(\theta, \phi),$$

$Y_{\ell m}$ the standard spherical harmonics. $\Delta Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$

Equation on $u_\ell(r,t)$:

$$\left[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - V_\ell(r)\right] u_\ell(r,t) = 0$$

$$r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right| + const$$

Perturbation of black holes

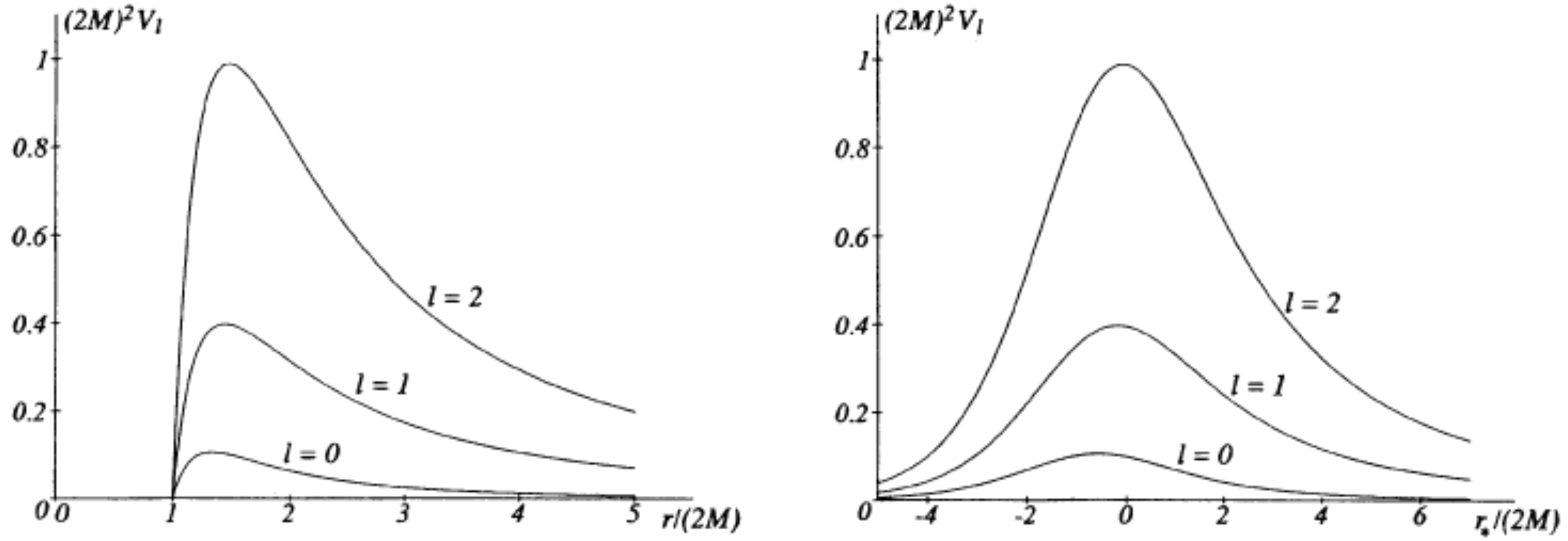
Test scalar field in the Schwarzschild geometry

Assume a harmonic time dependence, $u_\ell(r,t) = \hat{u}_\ell(r,\omega)e^{-i\omega t}$

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_\ell(r)\right]\hat{u}_\ell(r,\omega) = 0.$$

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right],$$

The effective potential $V_\ell(r)$ corresponds to a single potential barrier.



(a) (b)

Perturbation of black holes

Test scalar field in the Schwarzschild geometry

The potential vanishes at both $r = +\infty$ and $r = 2M \Rightarrow$

$$\hat{u}_\ell(r, \omega) \sim e^{\pm i\omega r*}$$

Two linearly independent solutions asymptotically.

Perturbation of black holes

The perturbed Schwarzschild metric can be written in general as

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}\left(d\phi - q_1 dt - q_2 dr - q_3 d\theta\right)^2 + e^{2\mu_2}dr^2 + e^{2\mu_3}d\theta^2$$

Where for the unperturbed case

$$e^{2\nu} = e^{-2\mu_2} = 1 - \frac{2M}{r},$$

and

$$e^{\mu_3} = r, \quad e^{\psi} = r\sin\theta, \quad q_1 = q_2 = q_3 = 0.$$

- Axial (odd-parity) perturbation:
  q_1, q_2, and q_3 are first-order quantities, introduce frame dragging (rotation of BH);

- Polar (even-parity) perturbation:
  For perturbations of $\delta\nu, \delta\psi, \delta\mu_2$, and $\delta\mu_3$ there is no frame dragging since $\phi \to -\phi$ is a symmetry for these perturbations.

Perturbation of black holes

$$h_{\mu\nu} = e^{-i\omega t}\tilde{h}_{\mu\nu}$$

In the Regge-Wheeler gauge

Axial perturbations

$$\tilde{h}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{bmatrix} \left(\sin\theta \frac{\partial}{\partial\theta}\right) Y_{l0}(\theta)$$

Polar perturbations

$$\tilde{h}_{\mu\nu} = \begin{bmatrix} H_0(r)f & H_1(r) & 0 & 0 \\ H_1(r) & H_2(r)/f & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2K(r)\sin^2\theta \end{bmatrix} Y_{l0}(\theta)$$

Perturbation of black holes

The Einstein equations give 10 coupled second-order differential equations: 3 for the odd radial variables, and 7 for the even variables.

- Odd perturbations they be combined in a single Regge-Wheeler gravitational variable $\Psi^{-}_{s=2}$,

- Even perturbations can be combined in a single Zerilli gravitational variable $\Psi^{+}_{s=2}$.

They satisfy the Schrödinger-like equation

$$\frac{d^2\Psi_s}{dr_*^2} + \left(\omega^2 - V_s\right)\Psi_s = 0$$

with the potentials

$$V^{-}_{s=2} = f(r)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]$$

and

$$V^{+}_{s=2} = \frac{2f(r)}{r^3}\frac{9M^3 + 3\lambda^2 M r^2 + \lambda^2(1+\lambda)r^3 + 9M^2\lambda r}{(3M+\lambda r)^2}.$$

$\lambda \equiv (l-1)(l+2)/2$

Boundary conditions

Boundary conditions at the horizon. The potential $V \to 0$ as $r_* \to -\infty$, so $\Psi \sim e^{-i\omega(t \pm r_*)}$. Nothing should leave the horizon: only ingoing modes (corresponding to a plus sign) should be present

$$\Psi \sim e^{-i\omega(t+r_*)}, \quad r_* \to -\infty\,(r \to r_+)\,.$$

Boundary conditions at spatial infinity. The potential is zero at infinity. Requiring

$$\Psi \sim e^{-i\omega(t-r_*)}, \quad r \to \infty,$$

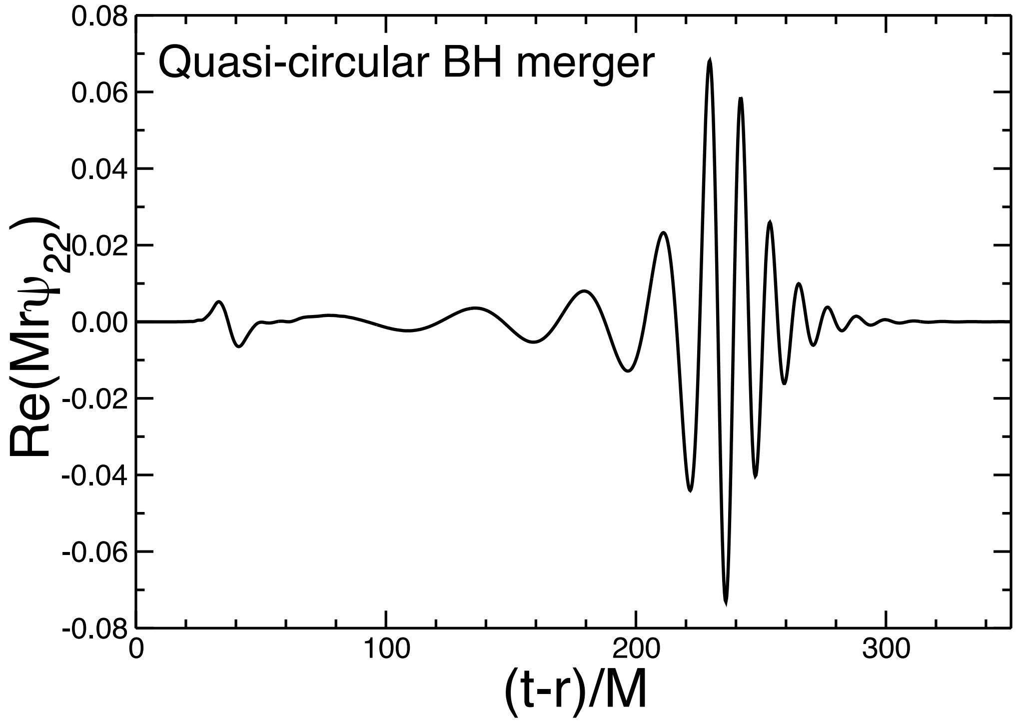
we discard unphysical waves "entering the spacetime from infinity". Only outgoing modes are allowed.

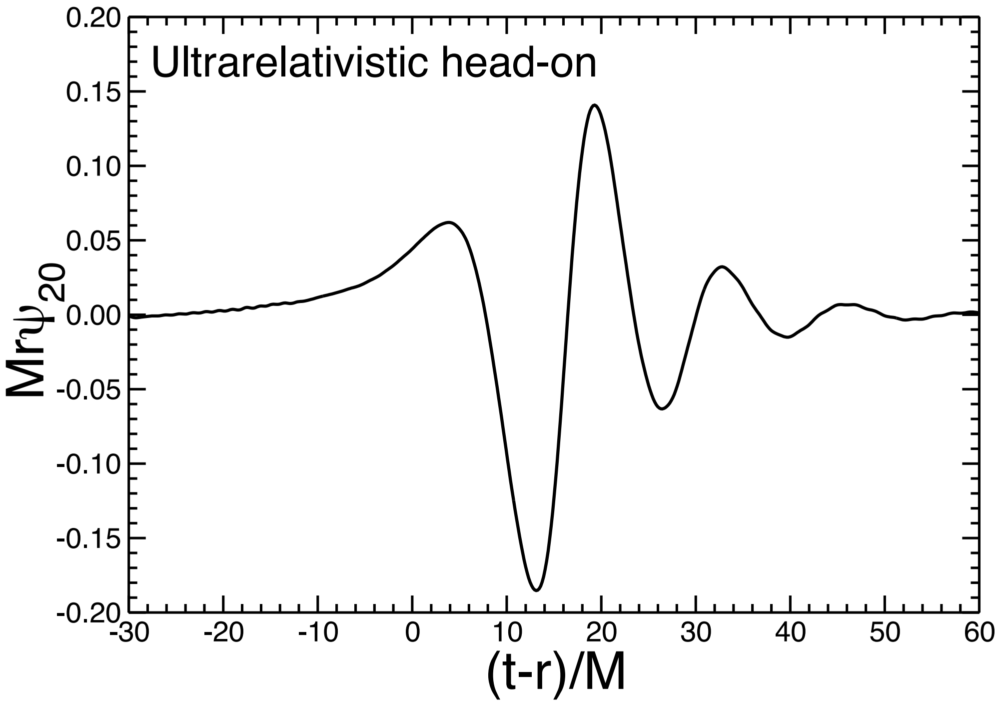
Quasinormal modes

- In this study waves escape either to infinity or into the BH.
- Difference between this type of QNM problems and other physical problems, i.e. the vibrating string: the system is now dissipative.
- For this reason an expansion in normal modes is not possible.
- There is a discrete infinity of QNMs, defined as eigenfunctions satisfying the boundary conditions.
- The corresponding eigenfrequencies ω_{QNM} have both a real and an imaginary part, the latter giving the (inverse) damping time of the mode.
- QNMs do not form a complete set of wavefunctions.

Ringing of a black hole leads to radiation of gravitational waves

Quasinormal modes



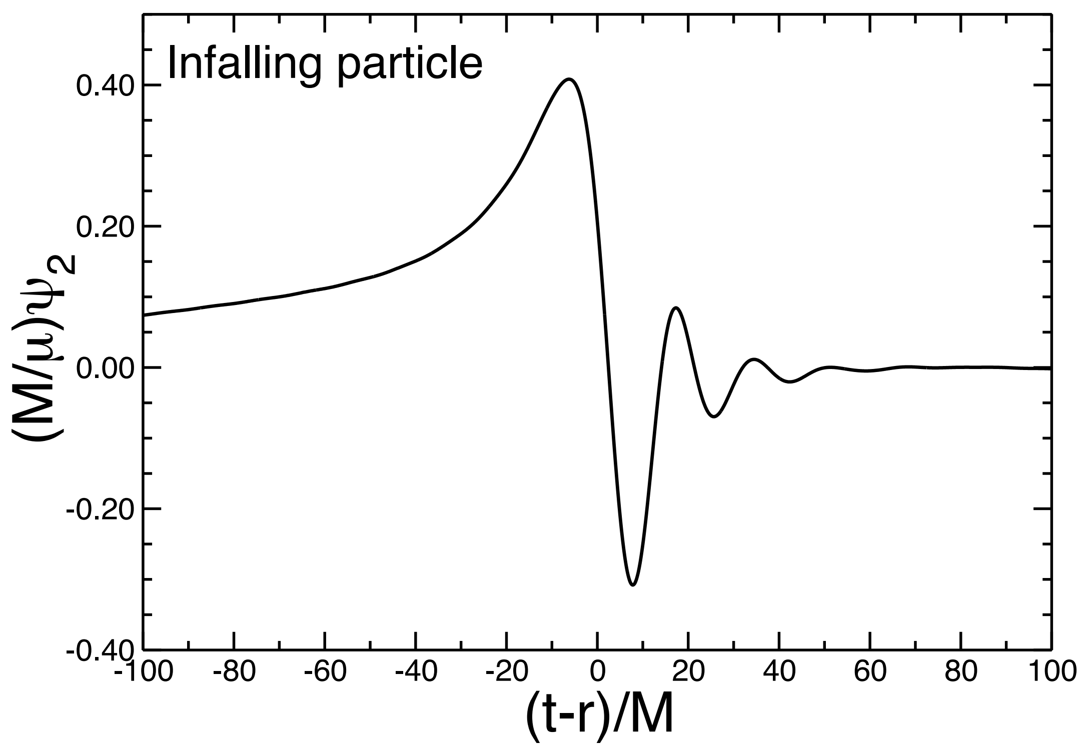


The signal from two equal-mass BHs initially on quasi-circular orbits, inspiralling towards each other due to the energy loss induced by gravitational wave emission, merging and forming a single final BH

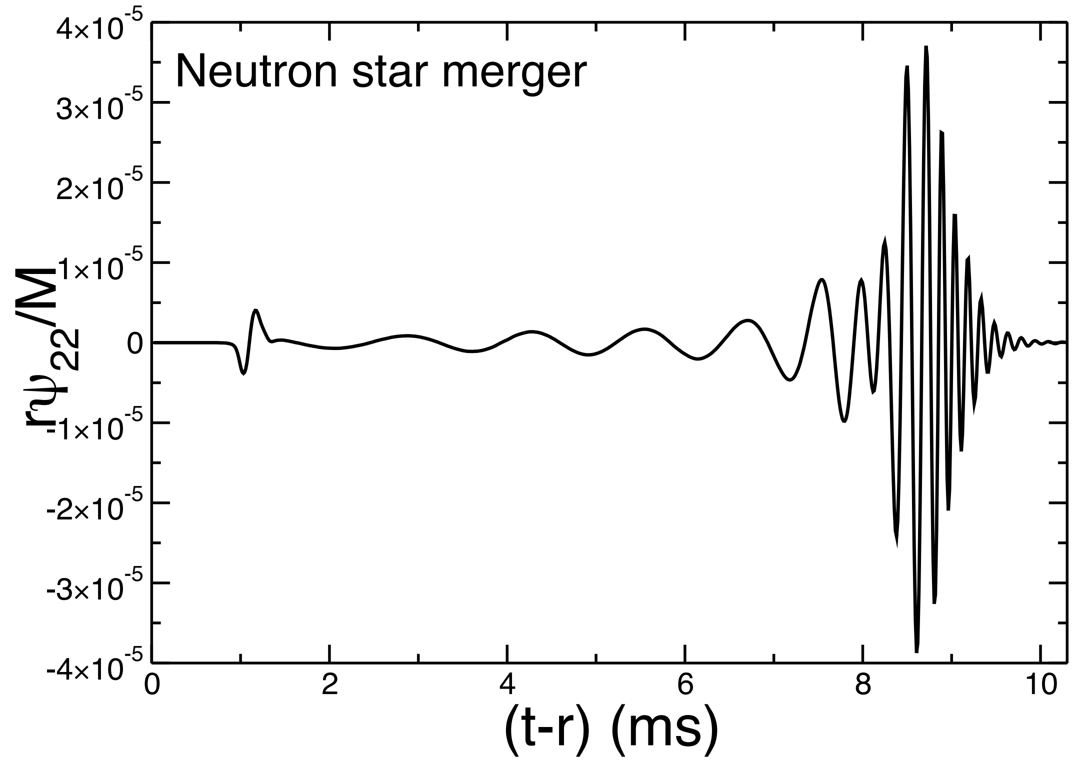
gravitational waveforms from numerical simulations of two equal-mass BHs, colliding head-on with $v/c = 0.94$

from E.Berti1, V.Cardoso, A.Starinets'09

Quasinormal modes



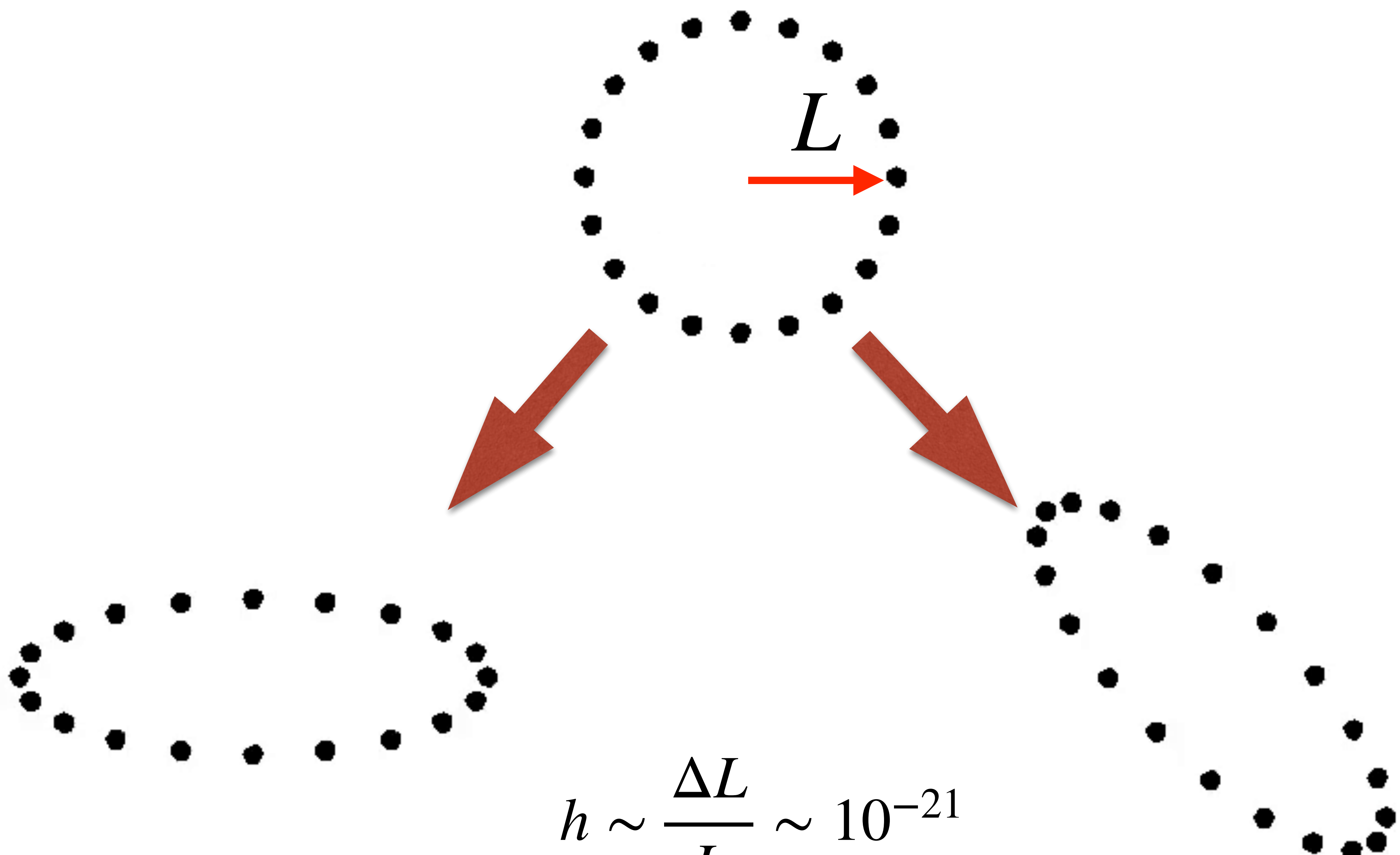
the gravitational waveform (or more precisely, the dominant, l = 2 multipole of the Zerilli function) produced by a test particle of mass μ falling from rest into a Schwarzschild BH



two massive neutron stars (NSs) with a polytropic equation of state, inspiralling and eventually collapsing to form a single BH.

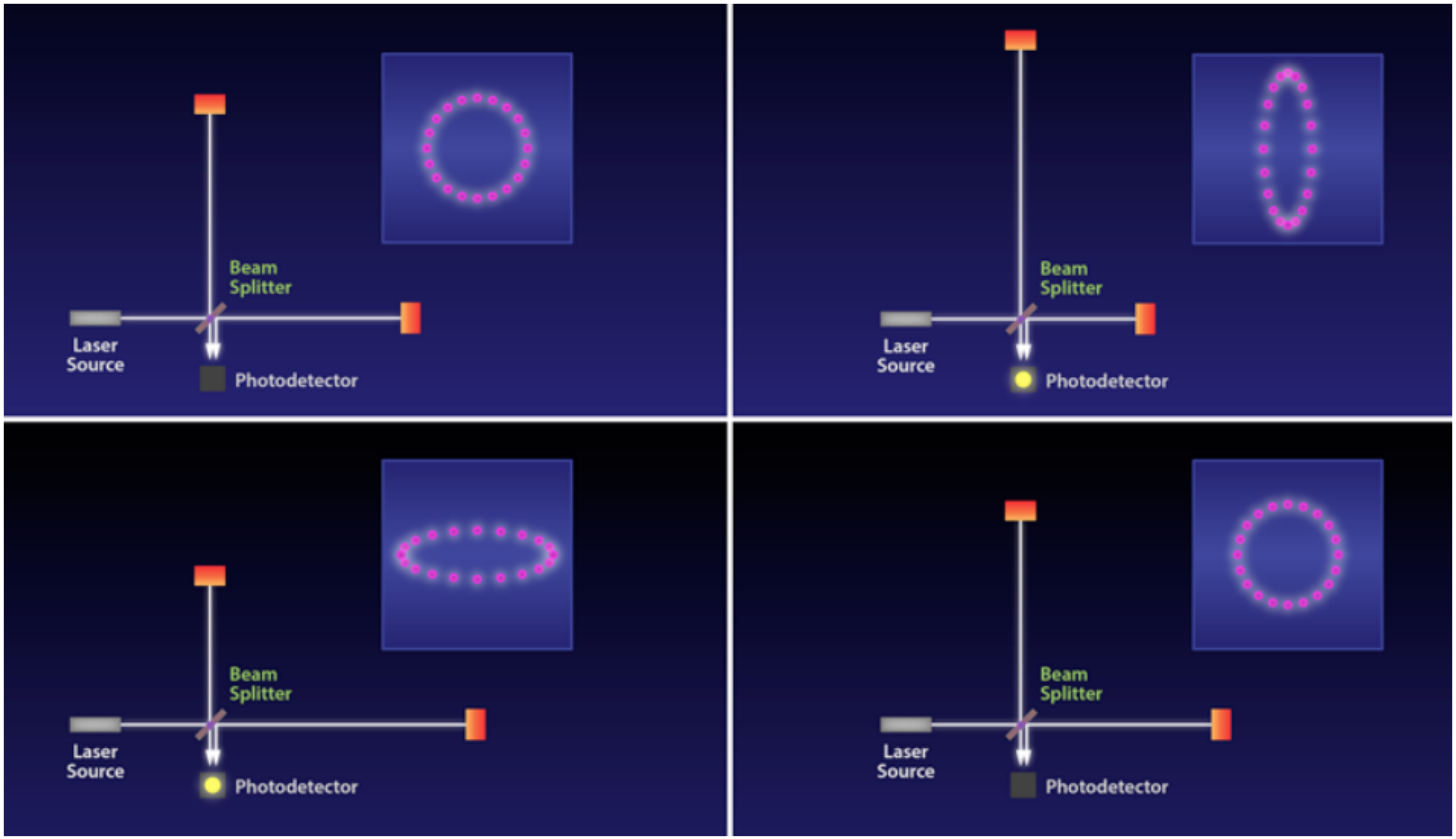
from E.Berti1, V.Cardoso, A.Starinets'09

Detection of gravitational waves



$$h \sim \frac{\Delta L}{L} \sim 10^{-21}$$

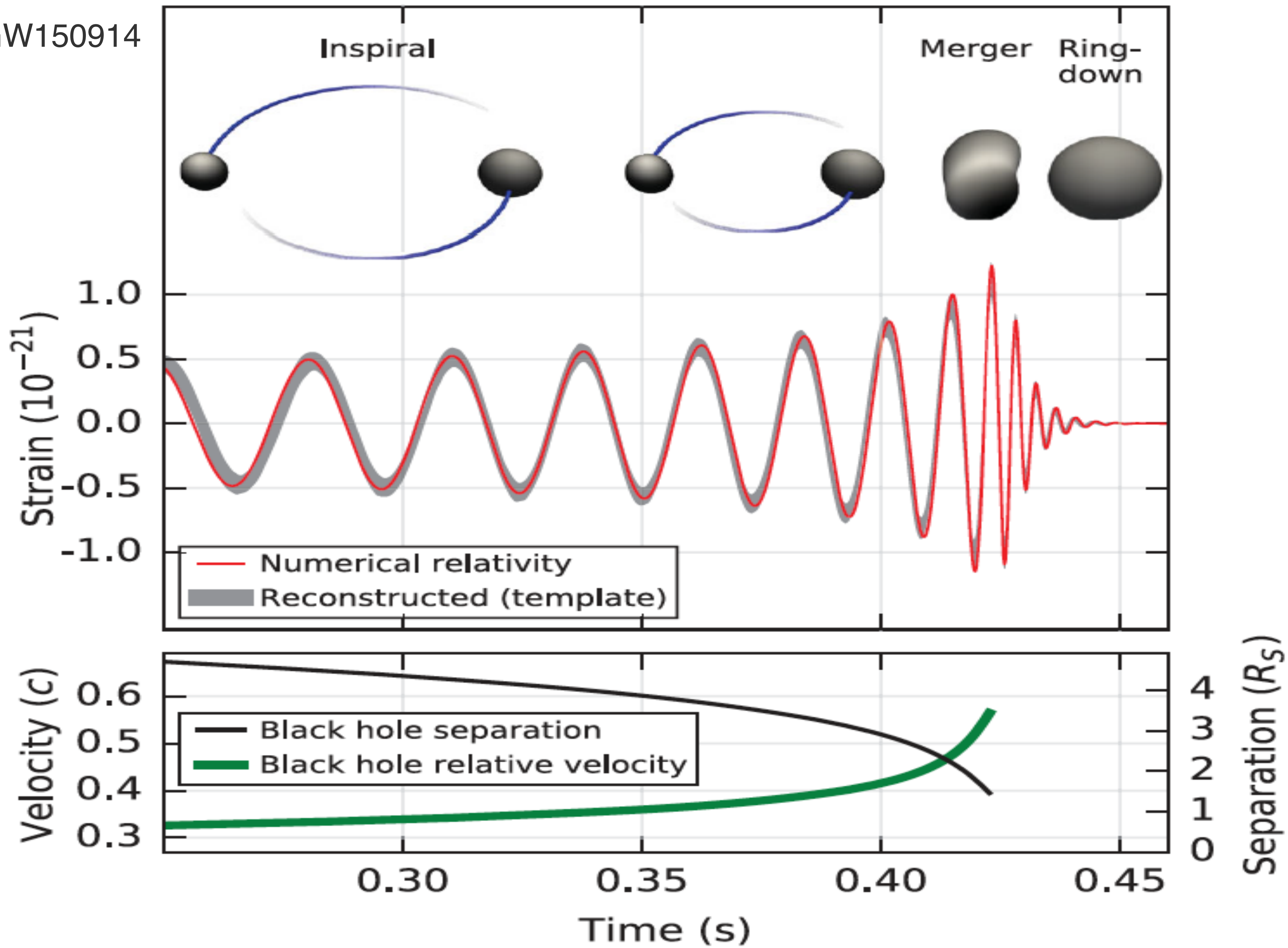
Gravitational wave interferometer



from talk by E.Porter

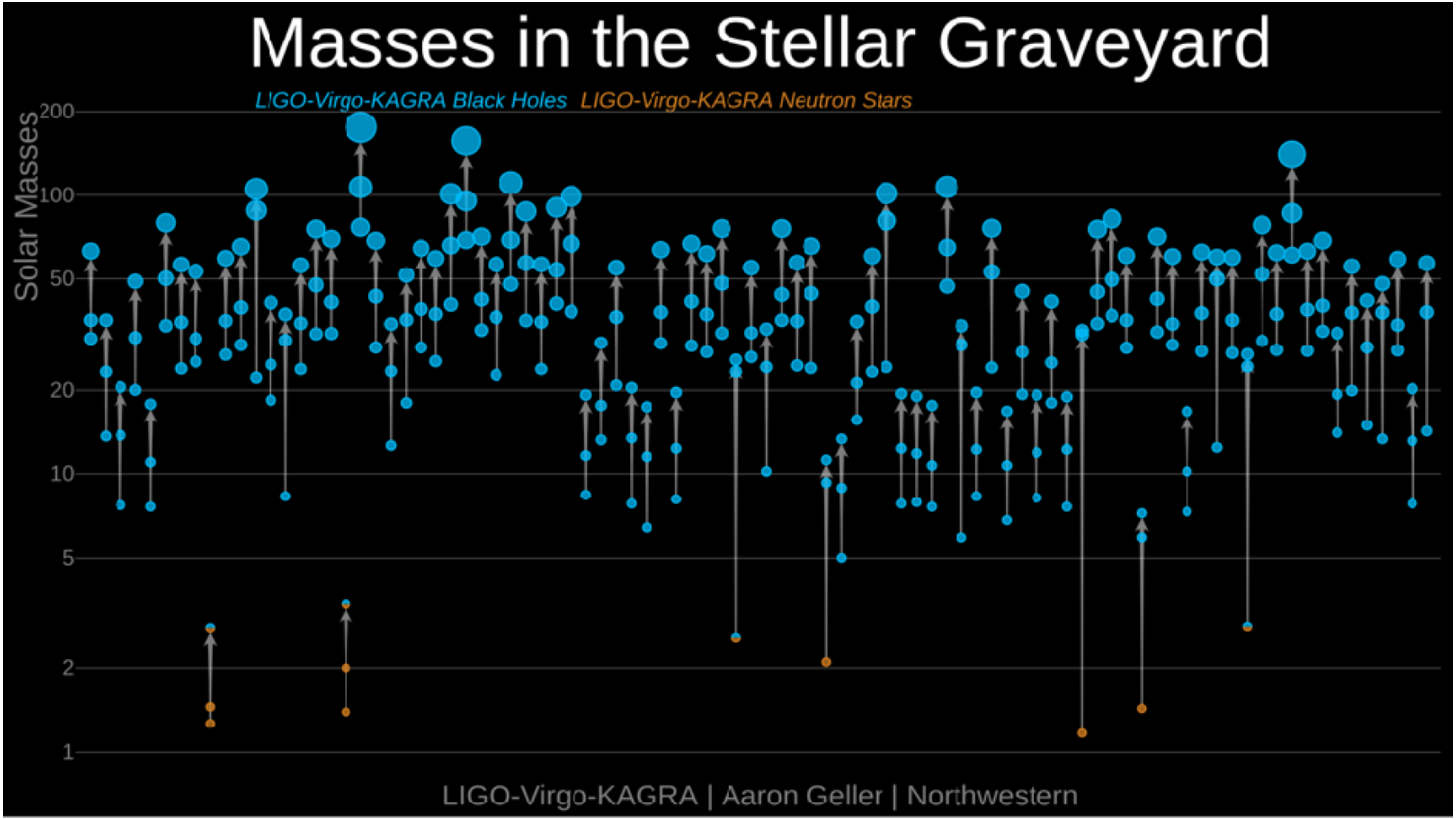
Gravitational wave interferometer

GW150914



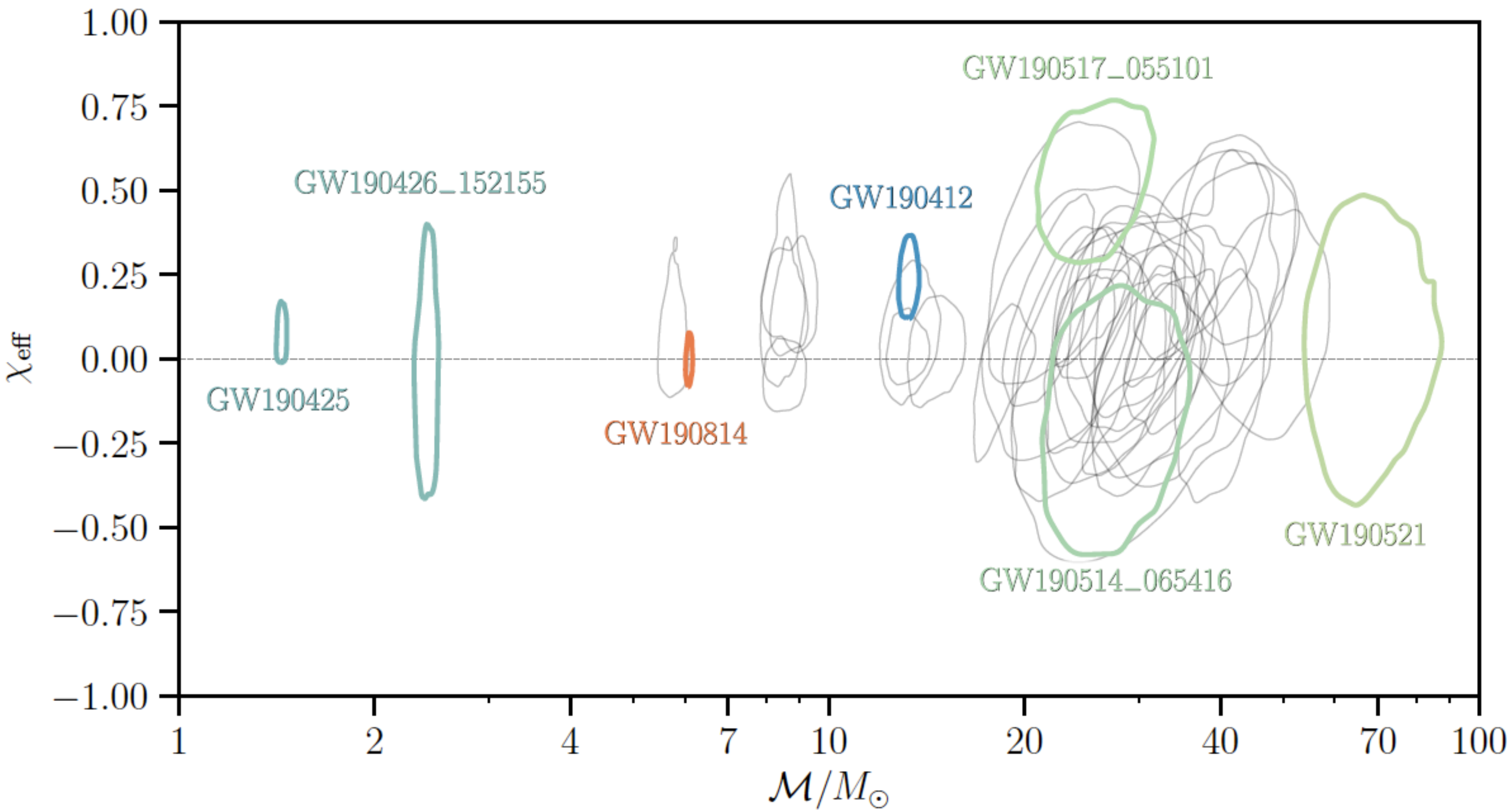
from talk by E.Porter

Gravitational wave interferometer



Masses of detected LIGO/Virgo compact binaries. This plot shows the masses of all compact binaries detected by LIGO/Virgo, with black holes in blue and neutron stars in orange. The objects are arranged in order of discovery date.

Gravitational wave interferometer



Effective spiral spin. Each contour represents the 90% credible region for a different event.

Abbott et al, arXiv:2010.14527(2020)

Constraints on gravity theories

- In general relativity $\omega = |k|$
- In modified gravity the relation can be different. E.g. in massive gravity
$$\omega^2 = k^2 + m_g^2$$
- Frequency dependent dephasing of of the GW signal.
- $$m_g \leq 1.76 \times 10^{-23}\ \mathrm{eV}/c^2$$

Constraints on gravity theories

GW170817

- Detected Aug. 17, 2017
- Brightest GW event yet seen
- Luminosity distance of 40 Mpc
- Multiple EM confirmations in gamma, x-ray, optical, radio

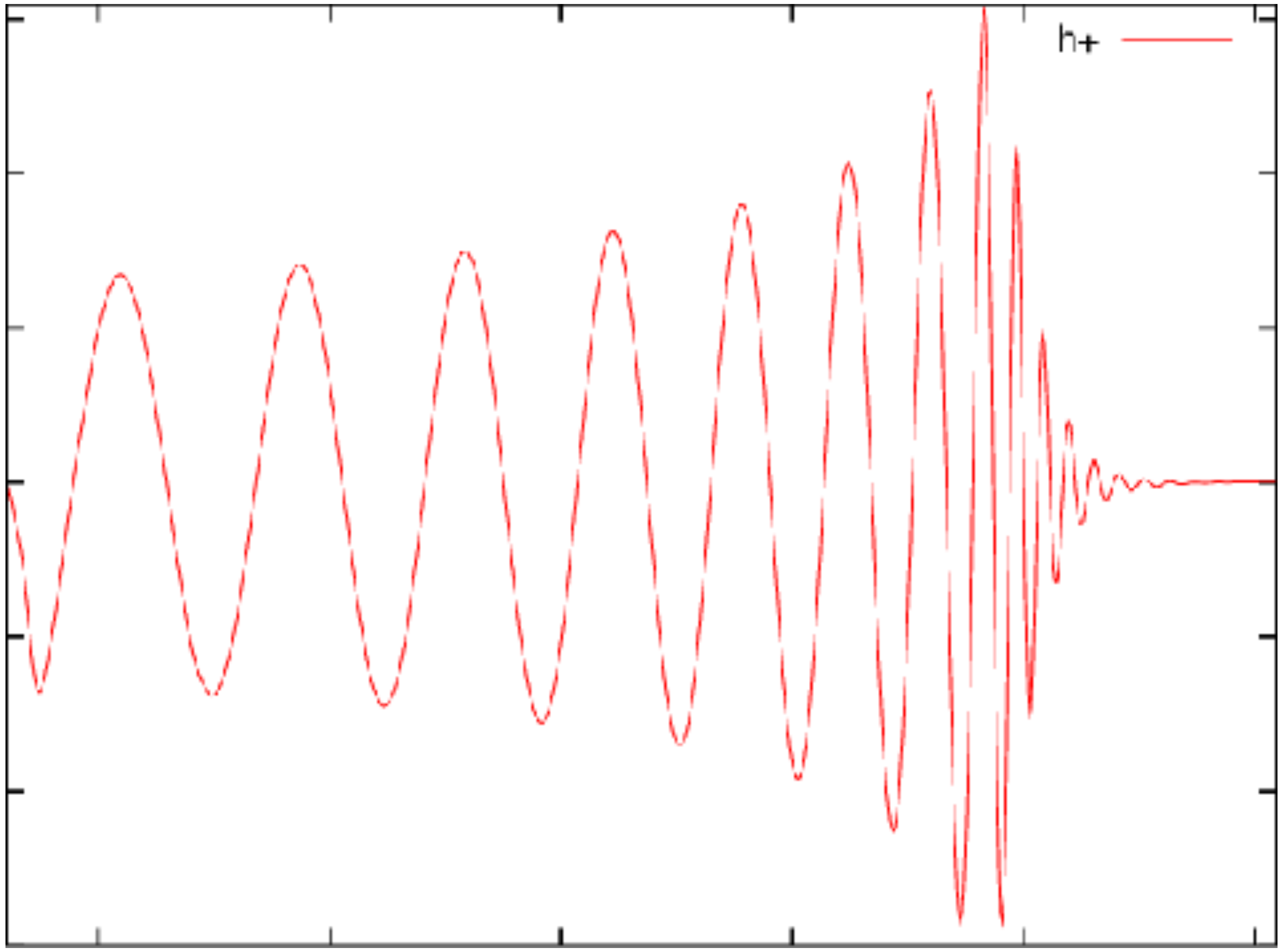
The time delay between the GW and GRB detections $\Delta t = (1.74 \pm 0.05)\, s$

Fractional difference between the speed of light and GWs $\dfrac{c_g - c}{c} \approx c\dfrac{\Delta t}{D_L}$

$$-3 \times 10^{-15} \leq \frac{\Delta c}{c} \leq 7 \times 10^{-16}$$

$\Rightarrow$ Many (solutions of) modified gravity theories are ruled out

Gravitational waves from black holes



Test GR
Rule out gravity theories
Get hints on modifications of gravity