Modern Trends in Mathematical Physics. II

(for pedestrians ... cyclists and drivers)

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Moscow International School of Physics 2022

JINR, Dubna, July 2022

Short summary

Mathematical physics:

- Liouville theory and 2d quantum gravity (with conformal matter);
- Generally: lack of naive continuation into higher dimensions;
- Partially possible – only for particular class of theories;
- Use of complexification, supersymmetry etc ...

Physical aims

Any physics?:

- Allows to address important physical questions:
  - Vacua solutions: number of/or space (moduli) of vacua ... branches ...;
  - Spectrum of (light?) excitations ... 'BPS-defended'
- Conjecture exact answers (physical intuition?);
- Mathematical 'proof' ... consistency ...;

Physical pendulum

EOM:

$$\ddot{q} + \Lambda^2 \sin q = 0, \qquad \Lambda^2 = \frac{g}{l} \tag{1}$$

- $\sin q \underset{q \simeq 0}{\simeq} q$ – mathematical pendulum (harmonic oscilator);
- Integrated from energy conservation

$$U = \frac{1}{2} p^2 - \Lambda^2 \cos q = \frac{1}{2} \dot{q}^2 - \Lambda^2 \cos q \tag{2}$$

with

$$t = \int \frac{dq}{p} = \frac{\partial}{\partial U} \int p dq \tag{3}$$

Integrability

- Integrable (as any) system with $\dim \mathcal{M} = 2$ and conserved energy (integral of motion!);
- Generalized to $\dim \mathcal{M} = 2 \cdot \#\mathrm{IOM}$ (Liouville-Arnold);
- Complexification: $(p, \exp(iq)) \subset (\lambda, w) \in \mathbb{C} \times \mathbb{C}^\times$

$$\Sigma: \ \Lambda^2 \left(w + \frac{1}{w} \right) = \lambda^2 - U$$

$$t \sim \int \frac{dw}{w\lambda} \sim \int \frac{d\lambda}{w - \frac{1}{w}}$$

(Elliptic) integral of a holomorphic differential on torus – elliptic curve Σ: (why – a problem for a seminar?)

Integrable systems

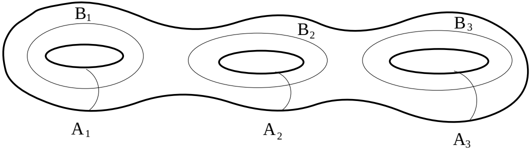
Integrability with $\#\mathrm{IOM} > 2$ is a very nontrivial property $(E = U, P, \dots)$

Problem (!?): for a system with Hamiltonian $H = \frac{1}{2}\sum_{i=1}^{3}\left(p_i^2 + \exp(q_{i+1} - q_i)\right)$ find all independent IOM.

- In practice: existence of Lax representation or overdetermined system of N^2 equations for $\dim\mathcal{M} = 2 \cdot N$ variables $\Rightarrow$ complexification;
- Toda systems: $L = p \cdot h + \sum_{\alpha\in\Pi} \exp(\alpha \cdot q)(e_\alpha + f_\alpha)$ (over simple roots of Lie algebras $\Rightarrow$ Lie groups);
- $\Sigma: \ 0 = \det(\lambda - L(w)) = P(\lambda) - w - \frac{1}{w}, \ P(\lambda)$ generates IOM;
- $\Sigma^{\otimes g} \subset \mathcal{T}^g$: linearization of dynamic.

Complex curves

Riemann surface ($g = 3$) Σ:



Smooth Riemann surface (of genus 3) with marked A- and B-cycles.

Lattice of charges $\Leftrightarrow$ $H_1(\Sigma)$ with symplectic $\langle , \rangle$, $\langle A_i, B_j \rangle = \delta_{ij}$.

Period matrix: $\mathrm{Im}\, T_{ij} \geq 0$, $T \underset{\text{degeneration}}{\rightarrow} \log a$

Period matrices

Computation of elliptic integrals $\oint \lambda \frac{dw}{w}$ or $\oint \frac{dw}{w\lambda}$ on $\Sigma:\ w + \frac{\Lambda^4}{w} = \lambda^2 - U$

At $\Lambda \to 0$, for $U = a^2$

$$w = \lambda^2 - a^2 \tag{4}$$

so that

$$\oint \lambda \frac{dw}{w} = \oint \lambda d \log\left(\lambda^2 - a^2\right) \sim \left\{ \begin{array}{c} a \\ a \log a - a \end{array} \right.$$

The period "matrix" $\tau \sim \frac{\partial}{\partial a}(a \log a - a) = \log a$, or

$$\tau \sim \int_{-a}^{a} d \log\left(\lambda^2 - a^2\right) \sim \log a \tag{5}$$

- To be identified with complexified gauge coupling $\tau \sim \frac{\vartheta}{2\pi} + i\frac{4\pi^2}{g^2}$;
- Any parallels with QFT?