

The branching fractions measurements of
 $J/\psi \rightarrow \pi^+\pi^-\eta$ and $J/\psi \rightarrow K^+K^-\eta$ at KEDR

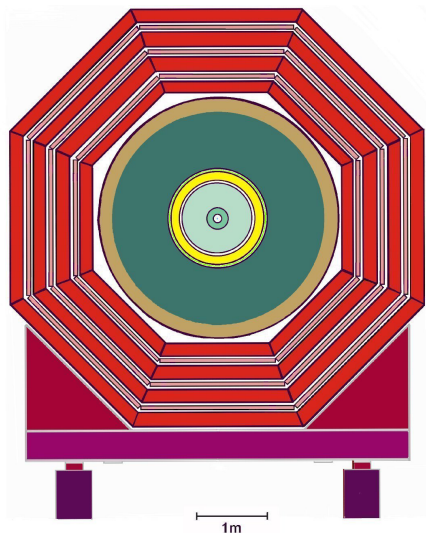
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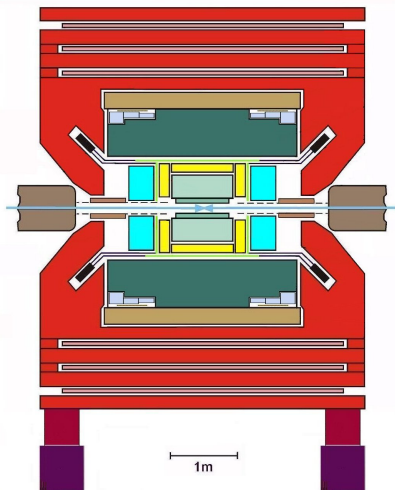
Moscow International School of Physics 2024 (MISP 2024)
28.02-06.03

The outline

- 1 The KEDR detector
- 2 The $J/\psi \rightarrow \pi^+ \pi^- \eta$ process
 - Idea of analysis
 - Selection criteria
 - Results
- 3 The $J/\psi \rightarrow K^+ K^- \eta$ process
 - Selection criteria
 - Results
- 4 Summary



The KEDR detector



The KEDR detector consists of:

- Vertex detector
- Drift chamber
- Aerogel counters
- Time of flight system
- LKr calorimeter
- CsI calorimeter
- Muon system

The $J/\psi \rightarrow \pi^+ \pi^- \eta$ process

The $J/\psi \rightarrow \pi^+\pi^-\eta$ process

- The $J/\psi \rightarrow \pi^+\pi^-\eta$ process has been measured by BaBar using ISR method and BES-III
- The dominant mod is a $J/\psi \rightarrow \rho\eta$ process, that was measured in 1988 и 1990 years at MARK-III and DM2 detectors

$$\text{BES-III: } (3.78 \pm 0.68) \times 10^{-4}$$

$$\text{BaBar: } (4.2 \pm 0.8) \times 10^{-4}$$

$$\text{PDG: } (3.8 \pm 0.7) \times 10^{-4}$$

$$J/\psi \rightarrow \pi^+\pi^-\eta$$

$$\text{MARK-III: } (1.93 \pm 0.13 \pm 0.29) \times 10^{-4}$$

$$\text{DM2: } (1.94 \pm 0.17 \pm 0.29) \times 10^{-4}$$

$$\text{PDG: } (1.93 \pm 0.23) \times 10^{-4}$$

$$J/\psi \rightarrow \rho\eta$$

$$Br(\rho \rightarrow \pi^+\pi^-) \simeq 100\%$$

$$Br(\eta \rightarrow \gamma\gamma) = (39.36 \pm 0.18)\%$$

There is a significant interference with

$J/\psi \rightarrow \omega\eta$ is expected

$$Br(\omega \rightarrow \pi^+\pi^-) = (1.53_{-0.13}^{+0.11})\% \text{ is}$$

small, but

$$Br(J/\psi \rightarrow \omega\eta) = (1.74 \pm 0.20) \times 10^{-3}$$

is 10 times bigger

Idea of analysis

$$\frac{d\sigma}{d\Gamma} = |a + be^{i\phi}|^2 = |a|^2 + |b|^2 + ab^*e^{-i\phi} + a^*be^{i\phi}$$

$$a = (p_{\pi^+} \times p_{\pi^-}) \sin\theta_n \frac{m_\rho^2}{q^2 - m_\rho^2 + iq\Gamma_\rho(q^2)} - \text{the decay amplitude}$$

$$\Gamma(q^2) = \Gamma \left(\frac{p_\pi(q^2)}{p_\pi(m_\rho^2)} \right)^3 \left(\frac{m_\rho^2}{q^2} \right) - \text{the decay width}$$

$$\begin{aligned} ab^*e^{-i\phi} + a^*be^{i\phi} = & \frac{2(p_{\pi^+} \times p_{\pi^-})^2 \sin^2\theta_n m_\rho^2 m_\omega^2 (q^4 + m_\rho^2 m_\omega^2 + q^2 \Gamma_\rho \Gamma_\omega)}{((q^2 - m_\rho^2)^2 + q^2 \Gamma_\rho^2) ((q^2 - m_\omega^2)^2 + q^2 \Gamma_\omega^2)} \cos\phi \\ & - \frac{2(p_{\pi^+} \times p_{\pi^-})^2 \sin^2\theta_n m_\rho^2 m_\omega^2 q^2 (m_\rho^2 + m_\omega^2)}{((q^2 - m_\rho^2)^2 + q^2 \Gamma_\rho^2) ((q^2 - m_\omega^2)^2 + q^2 \Gamma_\omega^2)} \cos\phi \\ & + \frac{2(p_{\pi^+} \times p_{\pi^-})^2 \sin^2\theta_n m_\rho^2 m_\omega^2 (q^3 \Gamma_\omega + q \Gamma_\rho m_\omega^2)}{((q^2 - m_\rho^2)^2 + q^2 \Gamma_\rho^2) ((q^2 - m_\omega^2)^2 + q^2 \Gamma_\omega^2)} \sin\phi \\ & - \frac{2(p_{\pi^+} \times p_{\pi^-})^2 \sin^2\theta_n m_\rho^2 m_\omega^2 (q^3 \Gamma_\rho + q \Gamma_\omega m_\rho^2)}{((q^2 - m_\rho^2)^2 + q^2 \Gamma_\rho^2) ((q^2 - m_\omega^2)^2 + q^2 \Gamma_\omega^2)} \sin\phi \\ N_{theor} = & N_\rho \varepsilon_\rho H_\rho + N_\omega \varepsilon_\omega H_\omega + \\ & + \sqrt{N_\rho N_\omega} (\varepsilon_{\cos^+} H_{\cos^+} - \varepsilon_{\cos^-} H_{\cos^-}) \cos(\phi) \\ & + \sqrt{N_\rho N_\omega} (\varepsilon_{\sin^+} H_{\sin^+} - \varepsilon_{\sin^-} H_{\sin^-}) \sin(\phi) \end{aligned}$$

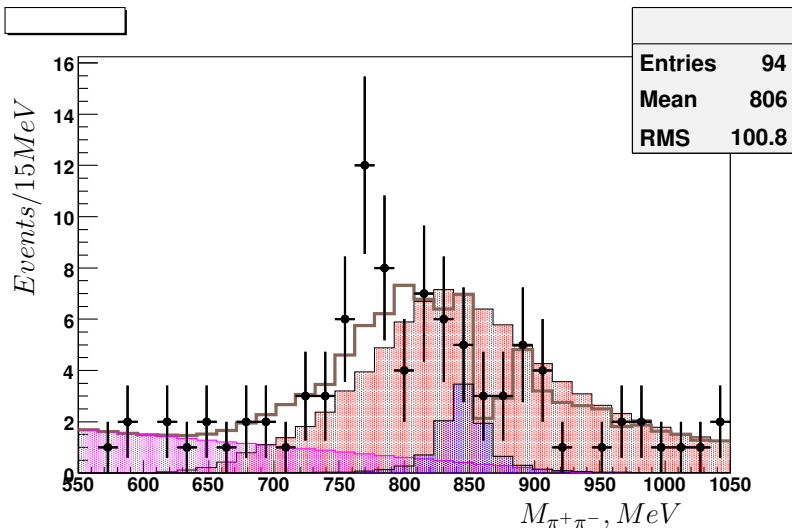
The selection was carried out using the likelihood function L :

$$L = -2 \sum (N_{th} - N_{exp} + N_{exp} \ln(N_{exp}/N_{th}))$$

- 1 $\chi^2 < 70$
- 2 $\chi^2 < \chi^2_{K+K-\eta}$
- 3 $(E_{\gamma_1} < 1300) \cap (E_{\gamma_2} > 200)$ — limit on the photons energy for cutting out the background from γf_0 and $\rho\pi$
- 4 $\cos(\theta_{\gamma\gamma}) > 0.2$ — limit on the angle between photons
- 5 $520 < M_{\gamma\gamma} < 580$ — limit on the η meson invariant mass
- 6 $-0.4 < \cos(\theta_{\pi^+\pi^-}) < 0.75$ — limit on the angle between pions

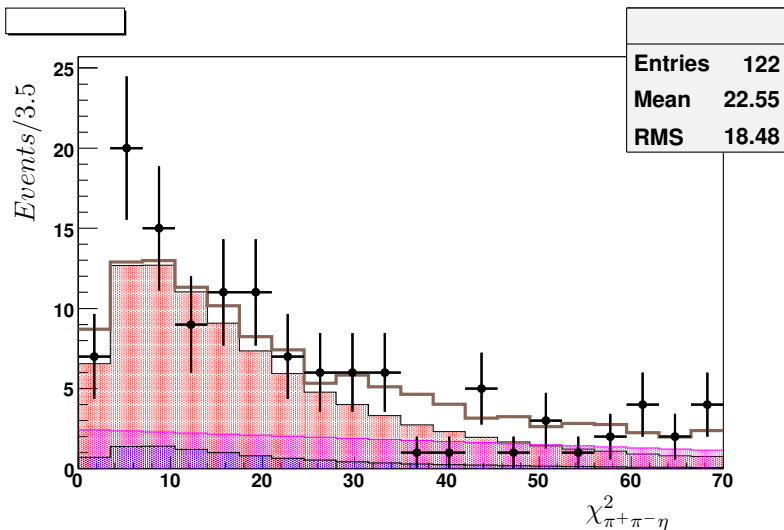
Efficiencies:

$$\varepsilon_\rho = 14.24 \pm 0.05\%, \quad \varepsilon_\omega = 14.82 \pm 0.05\%$$



The two pion invariant mass fit with interference ($\chi^2/Ndf = 15/25$), 15 MeV per bin, $\phi = (90.8 \pm 3.3 \pm 11.0)^\circ$

Results



The χ^2 distribution ($\chi^2/Ndf = 20/18$)

$$\text{KEDR: } Br(J/\psi \rightarrow \pi^+\pi^-\eta) = (3.77 \pm 0.54 \pm 0.34) \times 10^{-4}$$

$(N_{\pi^+\pi^-\eta} \approx 78)$

$$\text{KEDR: } Br(J/\psi \rightarrow \rho\eta) = (3.41 \pm 0.53 \pm 0.33) \times 10^{-4} \quad (N_\rho \approx 88)$$

$$\text{KEDR: } Br(J/\psi \rightarrow \omega\eta) = 2.352 \times 10^{-3} \text{ fixed} \quad (N_\omega \approx 10)$$

$$\text{BES-III: } (3.78 \pm 0.68) \times 10^{-4} \quad (N_{\pi^+\pi^-\eta} \approx 470)$$

$$\text{BaBar: } (4.2 \pm 0.8) \times 10^{-4} \quad (N_{\pi^+\pi^-\eta} \approx 50)$$

$$\text{PDG: } (3.8 \pm 0.7) \times 10^{-4}$$

$J/\psi \rightarrow \pi^+\pi^-\eta$

$$\text{MARK-III: } (1.93 \pm 0.13 \pm 0.29) \times 10^{-4} \quad (N_\rho \approx 100)$$

$$\text{DM2: } (1.94 \pm 0.17 \pm 0.29) \times 10^{-4} \quad (N_\rho \approx 300)$$

$$\text{PDG:}$$
$$(1.93 \pm 0.23) \times 10^{-4}$$

$J/\psi \rightarrow \rho\eta$

$$\text{BES-II: } (2.352 \pm 0.273) \times 10^{-3} \quad (N_\omega \approx 1250)$$

$$\text{BaBar: } (3.0 \pm 1.3 \pm 0.5) \times 10^{-3} \quad (N_\omega \approx 50)$$

$$\text{PDG: } (1.74 \pm 0.2) \times 10^{-3}$$

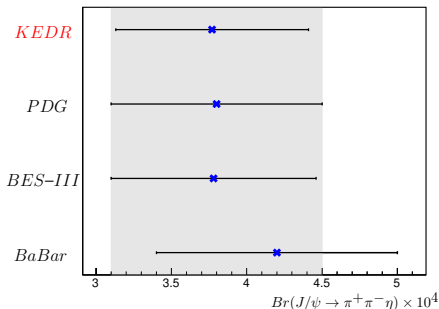
$J/\psi \rightarrow \omega\eta$

$$\text{MARK-III: } (1.71 \pm 0.08 \pm 0.2) \times 10^{-3} \quad (N_\omega \approx 70)$$

$$\text{DM2: } (1.43 \pm 0.1 \pm 0.21) \times 10^{-3} \quad (N_\omega \approx 100)$$

Result for $Br(J/\psi \rightarrow \pi^+\pi^-\eta)$

$$Br(J/\psi \rightarrow \pi^+\pi^-\eta) = (3.77 \pm 0.54 \pm 0.34) \times 10^{-4} \quad (N_{\pi^+\pi^-\eta} \approx 78)$$



The result comparisons for $Br(J/\psi \rightarrow \pi^+\pi^-\eta) \times 10^4$

Source	Unsert., %
Reconstruction	0.8
Fit var.	0.77
$Br(J/\psi \rightarrow \omega\eta)$	3.31
Simulation	0.76
Amount of J/ψ	1.1
$Br(\eta \rightarrow \gamma\gamma)$	0.46
Efficiency	0.77
Physical bg.	2.16
Selection criteria	8.12
Statistics	14.21
Sum	17

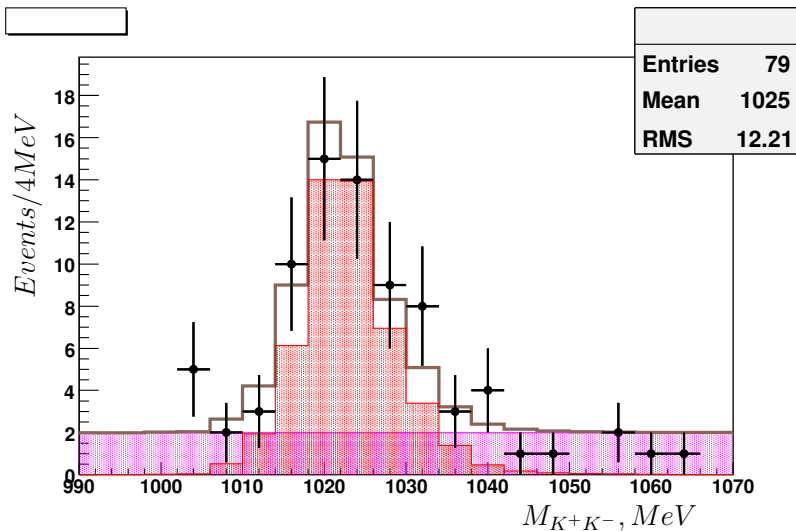
The $J/\psi \rightarrow K^+ K^- \eta$ process

The selection was carried out using the likelihood function L :

$$L = -2 \sum (N_{th} - N_{exp} + N_{exp} \ln(N_{exp}/N_{th}))$$

- 1 $\chi^2 < 70$
- 2 $\chi^2 < \chi^2_{\pi^+\pi^-\eta}$
- 3 $\sqrt{(E'_{K^+} - M_\phi/2)^2 + (E'_{K^-} - M_\phi/2)^2} < 10$ — after kinematic rec. (energy of K in the rest frame of ϕ meson)
- 4 $450 < M_\eta < 650$ — limit on the η meson mass before kinematic reconstruction

$$\text{Efficiency} - \varepsilon_{\phi\eta} = (7.62 \pm 0.08)\%$$

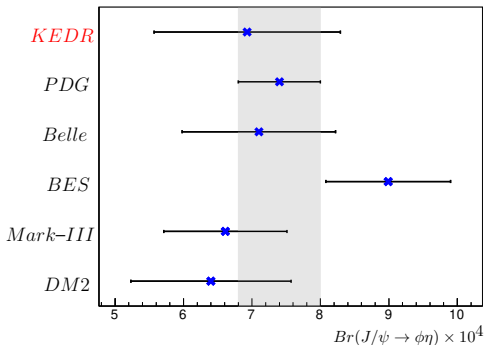


The two photons recoil invariant mass fit ($\chi^2/Ndf = 9/12$), 4 MeV per bin

Result for $Br(J/\psi \rightarrow K^+K^-\eta)$

$$Br(J/\psi \rightarrow \phi\eta) = (6.93 \pm 1.25 \pm 0.40) \times 10^{-4} \quad (N_\phi = 49)$$

$$Br(J/\psi \rightarrow K^+K^-\eta) = (3.40 \pm 1.25 \pm 0.40) \times 10^{-4}$$



The result comparison for
 $Br(J/\psi \rightarrow \phi\eta) \times 10^4$

Source	Unsert., %
Reconstruction	0.8
Fit	0.03
Simulation	0.62
Amount of J/ψ	1.1
$Br(\eta \rightarrow \gamma\gamma)$	0.46
$Br(\phi \rightarrow K^+K^-)$	1.02
Efficiency	0.9
Phys. bg.	1.95
Selection criteria	5.1
Statistics	18.1
Sum	19

- The branching measurement accuracy for $J/\psi \rightarrow \pi^+\pi^-\eta$ and $J/\psi \rightarrow \phi\eta$ is comparable to previous measurements
- Branching for the processes $J/\psi \rightarrow \rho\eta$ have a large uncertainties and is in poor agreement with previous measurements

Measurement results:

$$Br(J/\psi \rightarrow \rho\eta) = (3.41 \pm 0.53 \pm 0.33) \times 10^{-4}$$

$$Br(J/\psi \rightarrow \pi^+\pi^-\eta) = (3.77 \pm 0.54 \pm 0.34) \times 10^{-4}$$

$$Br(J/\psi \rightarrow \phi\eta) = (6.93 \pm 1.25 \pm 0.39) \times 10^{-4}$$

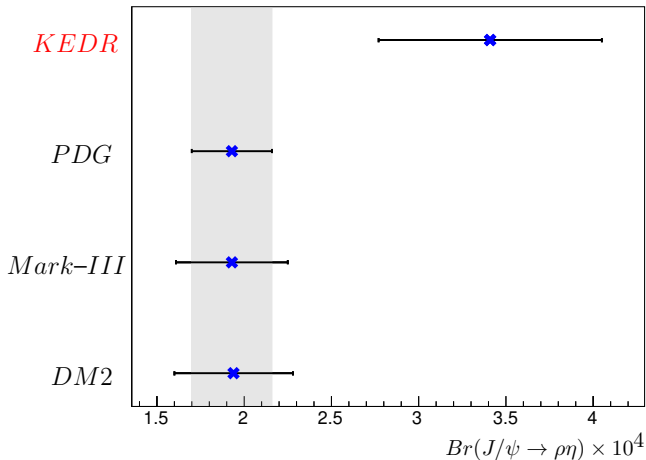
$$Br(J/\psi \rightarrow K^+K^-\eta) = (3.40 \pm 1.25 \pm 0.39) \times 10^{-4}$$

Thank you for your attention!

Back Up

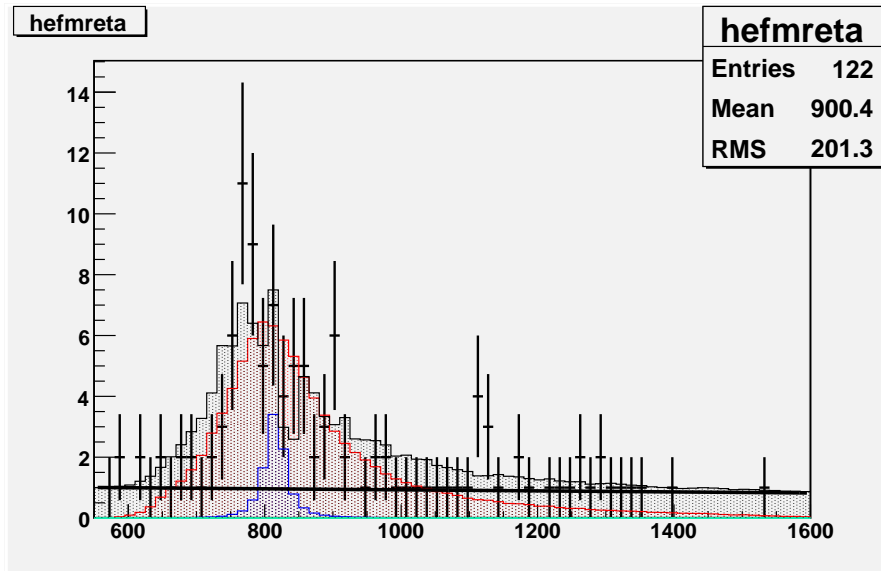
Result for $Br(J/\psi \rightarrow \rho\eta)$

$$Br(J/\psi \rightarrow \rho\eta) = (3.41 \pm 0.53 \pm 0.33) \times 10^{-4} \quad (N_\rho \approx 88)$$



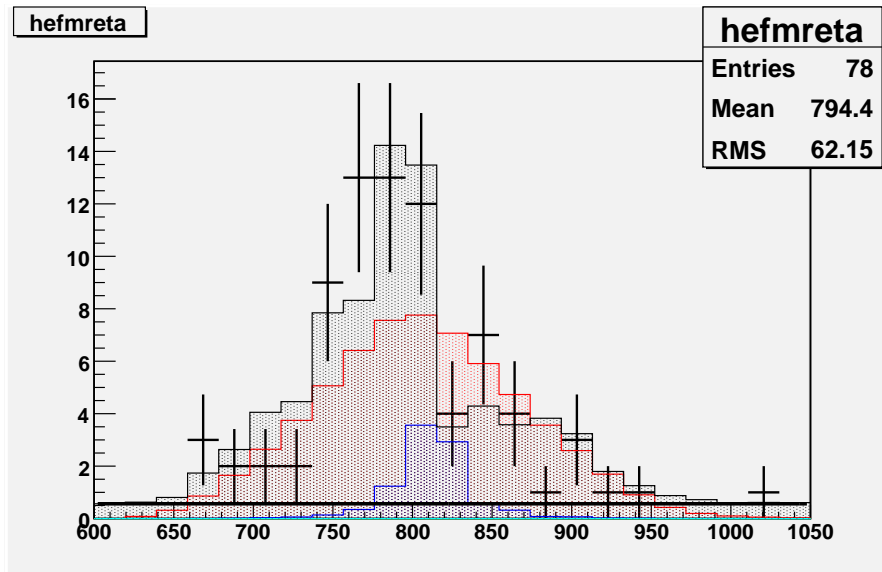
The result comparisons for $Br(J/\psi \rightarrow \rho\eta) \times 10^4$

Results for $\pi^+\pi^-\eta$



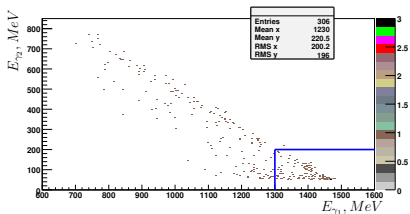
Pion invariant mass up to 1.6 GeV

Results for $\pi^+\pi^-\eta$

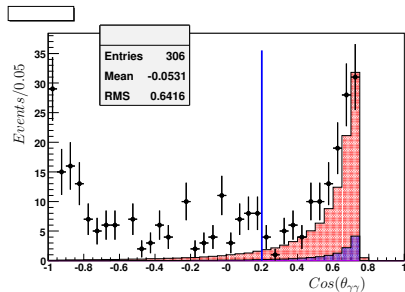


Pion invariant mass when $M_{\gamma\gamma}$ fixed at M_η

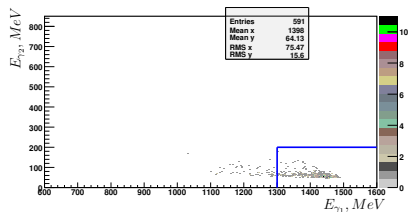
Contribution from $\rho\pi$



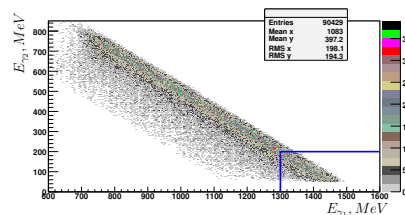
Photons energies. Experiment



Cosine between two photons

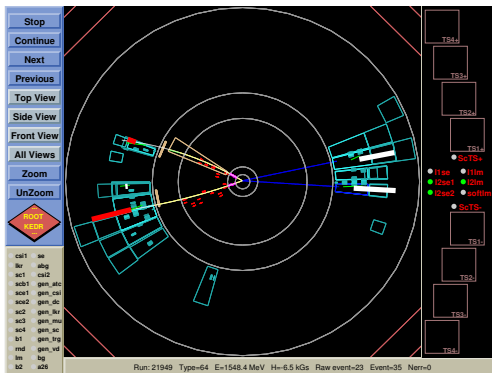


Photons energies. Simulation $\rho\pi$



Photons energies. Simulation $\pi^+\pi^-\eta$

Results for $\pi^+\pi^-\eta$



Example of $\pi^+\pi^-\eta$ reconstruction in the detector

Efficiencies:

$$\begin{aligned} \varepsilon_\rho &= 14.24 \pm 0.05\% & \varepsilon_\omega &= 14.82 \pm 0.05\% \\ \varepsilon_{\cos^+} &= 14.88 \pm 0.05\% & \varepsilon_{\cos^-} &= 14.85 \pm 0.05\% \\ \varepsilon_{\sin^+} &= 14.68 \pm 0.05\% & \varepsilon_{\sin^-} &= 14.89 \pm 0.05\% \end{aligned}$$

Results for fixed $Br(J/\psi \rightarrow \omega\eta) = 2.352 \times 10^{-3}$ from BES-II result:

- 1 $Br(J/\psi \rightarrow \rho\eta) = (3.41 \pm 0.53 \pm 0.33) \times 10^{-4}$
- 2 $Br(J/\psi \rightarrow \omega\eta) = 2.352 \times 10^{-3}$ *fixed*
- 3 $\phi = (90.8 \pm 3.3 \pm 11.0)^\circ$
- 4 $\Delta M = (62.8 \pm 12.6 \pm 48.0) \text{ M}\bar{\Delta}\text{B}$

The systematic uncertainties in $Br(J/\psi \rightarrow \pi^+\pi^-\eta)$,
selection criteria

Cut	Var.	$\Delta N/N, \%$	Unsert., %
$\chi^2 < 70$	$\chi^2 < 110$	7	3.77
$\chi^2 < \chi^2_{K+K-\eta}$	—	28	2.06
$E_{\gamma_1} < 1300; E_{\gamma_2} > 200$	—	12	0.43
$\cos(\theta_{\gamma\gamma}) > 0.2$	—	2	4.00
$520 < M_{\gamma\gamma} < 580$	$480 < M_{\gamma\gamma} < 620$	24	5.45
$-0.4 < \cos(\theta_{\pi^+\pi^-}) < 0.75$	—	6	1.23
Sum	—	—	8.12

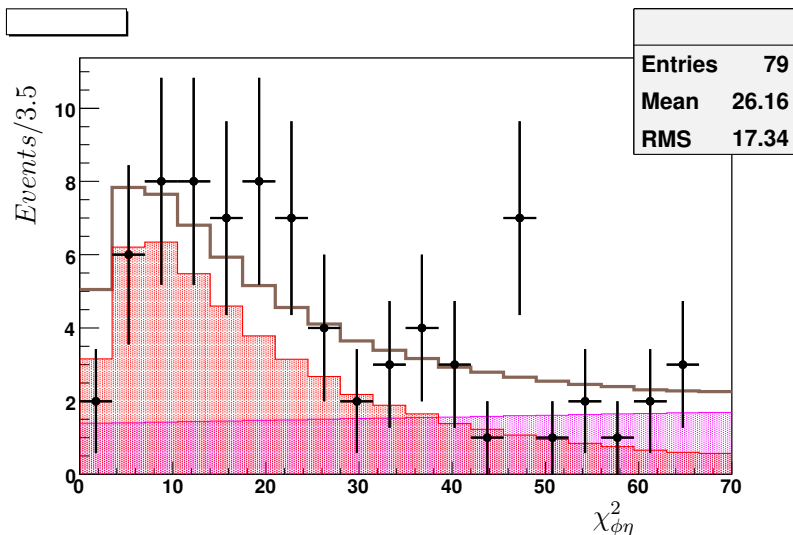
The systematic uncertainties in $Br(J/\psi \rightarrow \pi^+\pi^-\eta)$

Source	Unsert., %
Track reconstruction	0.5
η reconstruction	0.2
p/θ resolution	0.5
Nuclear interaction	0.4
Sum	0.8

Source	Unsert., %
Fit var.	0.77
Efficiency	0.77
$Br(J/\psi \rightarrow \omega\eta)$	3.31

Source	Unsert., %
Γ_ρ	0.024
M_ρ	0.44
Γ_ω	0.44
M_ω	0.43
Sum	0.76

Source	Unsert., %
$K^+K^-\eta$	0.53
$K_s K_l \eta$	0.12
$K_s K^* \cap K_s \overline{K^*}$	1.40
$\rho\pi$	0.70
$\rho'\pi$	0.78
$\gamma f_0(500)$	0.36
$\gamma f_2(1270)$	0.64
$\pi^+\pi^-\pi_0\eta$	0.87
Sum	2.16



The fit χ^2 ($\chi^2/Ndf = 15/17$)

① $Br(J/\psi \rightarrow \phi\eta) = (6.93 \pm 1.25 \pm 0.40) \times 10^{-4}$ ($N_\phi = 49$)

② $\Delta M = (2.07 \pm 1.04 \pm 0.57)$ MeV

Belle: $(7.1 \pm 1.0 \pm 0.5) \times 10^{-4}$ ($N_{\phi\eta} \approx 99$, 2023)

BES: $(8.99 \pm 0.18 \pm 0.89) \times 10^{-4}$ (2005)

DM2: $(6.4 \pm 0.4 \pm 1.1) \times 10^{-4}$ ($N_{\phi\eta} \approx 346$, 1990)

MARK-III: $(6.61 \pm 0.45 \pm 0.78) \times 10^{-4}$ (1988)

PDG: $(7.4 \pm 0.6) \times 10^{-4}$
 $J/\psi \rightarrow \phi\eta$

The systematic uncertainties in $Br(J/\psi \rightarrow K^+K^-\eta)$,
selection criteria

Cut	Used.	Var.	$\Delta N/N, \%$	Unsert., %
χ^2	$\chi^2 < 70$	$\chi^2 < 100$	17	2.5
$\chi_{\pi^+\pi^-\eta}^2$	$\chi^2 < \chi_{\pi^+\pi^-\eta}^2$	-	3	2.5
$E_{K^{+-}}$	$E_{K^{+-}} < 10$	$E_{K^{+-}} < 20$	20	1.3
M_η	$450 < M_\eta < 650$	$M_\eta < 700$	5	3.4
Sum	-	-	-	5.1

The systematic uncertainties in $Br(J/\psi \rightarrow K^+K^-\eta)$

Source	Unsert., %
Track Reconstruction	0.5
η reconstruction	0.2
p/θ resolution	0.5
Nuclear interaction	0.4
Sum	0.8

Source	Unsert., %
Γ_ϕ	0.44
M_ϕ	0.43
Sum	0.62

Source	Unsert., %
Fit var.	0.03
Efficiency	0.9

Source	Unsert., %
$\pi^+\pi^-\eta$	0.20
$K_S K^* \cap K_S \bar{K}^*$	1.36
$\rho\pi$	1.11
$\pi^+\pi^-\pi_0\eta$	0.82
Sum	1.95

KEDR detector parameters

Drift chamber:

- Inner radius: 125 mm
- Outer radius: 535 mm
- Length: 1100 мм
- Amount of axial superlayers: 4
- Amount of stereo superlayers: 3
- Amount of measurements: 42
- Amount of cells: 252
- Spatial resolution: 150 мкм
- dE/dx : 8.2%

CsI calorimeter:

- Polar angle: (6 - 38) degrees
- Thickness: 30 cm ($15 X_0$)
- Energy resolution for 0.1 GeV: 3%
- Energy resolution for 1 GeV: 2.5%
- Angle resolution for 0.1 GeV: 18 mrad
- Angle resolution for 1 GeV: 9 mrad

LKr calorimeter:

- Polar angle: (38 - 142) degrees
- Inner radius: 75 cm
- Thickness: 68 cm ($14.8 X_0$)
- Energy resolution for 0.1 GeV: 6%
- Energy resolution for 1 GeV: 2.5%
- Angle resolution for 0.1 GeV: 4 mrad
- Angle resolution for 1 GeV: 4 mrad

Physical backgrounds

$$Br(\pi^+\pi^-\eta) = (3.8 \pm 0.8) \times 10^{-4}$$

$$Br(\rho\eta) = (1.93 \pm 0.23) \times 10^{-4}$$

$$Br(\omega\eta) = (1.74 \pm 0.2) \times 10^{-3}$$

$$Br(\rho\pi) = (1.69 \pm 0.15) \times 10^{-2}$$

$$Br(\phi\eta) = (7.4 \pm 0.8) \times 10^{-4}$$

$$Br(\rho(1450)\pi \rightarrow 3\pi) = (2.3 \pm 0.7) \times 10^{-3}$$

$$Br(\pi^+\pi^-\pi_0\eta) = (1.17 \pm 0.2) \times 10^{-2}$$

$$Br(\omega\pi_0) = (4.5 \pm 0.5) \times 10^{-3}$$

$$Br(\omega\pi_0\pi_0) = (3.4 \pm 0.8) \times 10^{-3}$$

$$Br(\omega\eta\pi_0) = (3.4 \pm 1.7) \times 10^{-4}$$

$$Br(\rho(1450)\eta' \rightarrow 2\pi\eta') = (3.3 \pm 0.7) \times 10^{-6}$$

$$Br(\rho \rightarrow \pi^+\pi^-) \approx 100\%$$

$$Br(\omega \rightarrow \pi^+\pi^-) = (1.53 \pm 0.13) \times 10^{-2}$$

$$Br(\eta \rightarrow \gamma\gamma) = (39.36 \pm 0.18) \times 10^{-2}$$

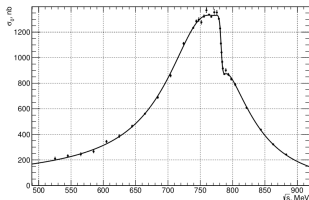
$$Br(\pi_0 \rightarrow \gamma\gamma) = (98.823 \pm 0.034) \times 10^{-2}$$

$$Br(\phi \rightarrow K^+K^-) = (49.1 \pm 0.5) \times 10^{-2}$$

$$Br(\eta' \rightarrow \gamma\gamma) = (2.307 \pm 0.033) \times 10^{-2}$$

SND (Novosibirsk) and BES-III (China)

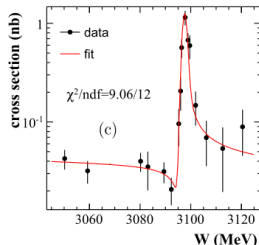
Achasov, M. N., et al. "Measurement of the $e^+e^- \rightarrow \pi^+\pi^-$ process cross section with the SND detector at the VEPP-2000 collider in the energy region $0.525 < \sqrt{s} < 0.883$ GeV." *Journal of High Energy Physics* 2021.1 (2021): 1-24.



$$|A_{\pi\pi}(s)|^2 = \left| \sqrt{\frac{3}{2}} \frac{1}{\alpha} \sum_{V=\rho,\omega,\rho'} \frac{\Gamma_V m_V^3 \sqrt{m_V} \sigma(V \rightarrow \pi^+\pi^-)}{D_V(s)} \frac{e^{i\phi_{\rho V}}}{\sqrt{q_\pi^3(m_V)}} \right|^2$$

$$\phi = (110.7 \pm 1.1 \pm 1.0)^\circ$$

Ablikim, M., et al. "Measurement of the phase between strong and electromagnetic amplitudes of J/ψ decays." *Physics Letters B* 791 (2019): 375-384.

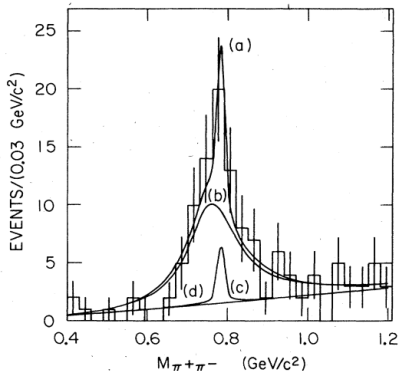


$$\sigma^0(W) = \left(\frac{A}{W^2} \right)^2 \frac{4\pi\alpha^2}{W^2}$$

$$\times \left| 1 + \frac{3W^2 \sqrt{\Gamma_{ee}\Gamma_{\mu\mu}} C_1 e^{i\Phi_{\gamma,\text{cont}}} (1 + C_2 e^{i\Phi})}{\alpha M(W^2 - M^2 + iM\Gamma)} \right|^2$$

$$\Phi(\text{fixed}) = 0^\circ$$

The MARK-III old work

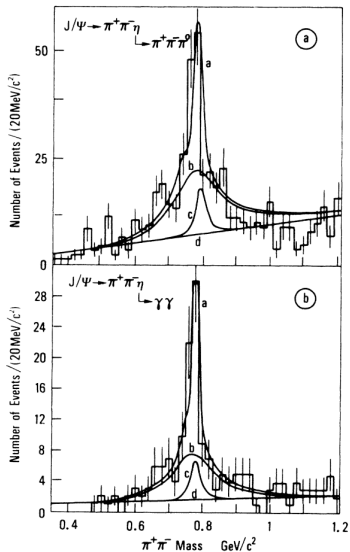
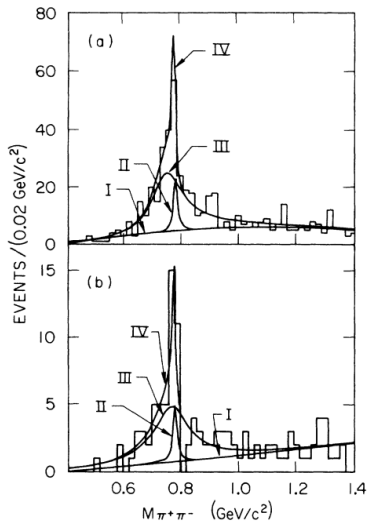


Fit 1	Fit 2
$N_\omega = 5.0 \pm 2.4$	$N_\omega = 8.9, \text{ fixed}$
$N_{\rho^0} = 58.5 \pm 7.4$	$N_{\rho^0} = 49.5 \pm 6.3$
$\phi = 0.4 \pm 0.5$	$\phi = 0.4 \pm 0.5$

$$\frac{d^3\sigma}{d \cos\theta_V d \cos\theta_1 d\varphi_1} \propto \sin^2\theta_1 [1 + \cos^2\theta_V + \sin^2\theta_V \cos(2\varphi_1)]$$

$$\begin{aligned} N_T &= N_{\rho^0} + N_\omega + 2 \left[\frac{\Gamma_\omega}{\Gamma_\rho} N_{\rho^0} N_\omega \right]^{1/2} \cos\phi \\ &= N_{\rho^0} + N_\omega + 0.507 (N_{\rho^0} N_\omega)^{1/2} \cos\phi . \end{aligned}$$

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$$F(m_{\pi^+\pi^-}) = |A_\rho(m_{\pi^+\pi^-}) + A_\omega(m_{\pi^+\pi^-}) e^{i\phi}|^2 + \text{linear background}.$$

$$A_V(m_{\pi^+\pi^-}) = \sqrt{N_V} F_{BW_V}(m_{\pi^+\pi^-}) \quad (V \equiv \omega, \rho) \quad / 16$$