Black holes

Part 1. Black holes in General Relativity

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Flat spacetime

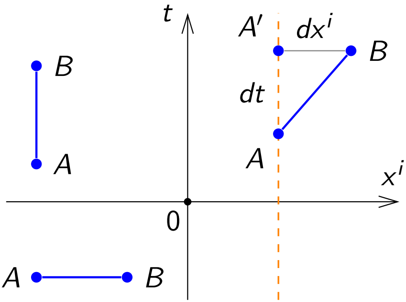
- Universal units: $\hbar = c = 1$
- Coordinates:

$$x^\mu = (t,\, x^1,\, x^2,\, x^3)$$

- Interval:

$$ds^2 = -dt^2 + (dx^i)^2$$

  - measures time & distance
  - Lorentz–invariant



- Gives causality:

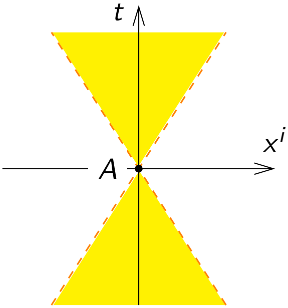
  → $ds^2 = 0$ — light-like

  → $ds^2 < 0$ — time-like

  $d\tau = \sqrt{-ds^2}$ — proper time

  → $ds^2 > 0$ — space-like

  $dl = \sqrt{ds^2}$ — distance



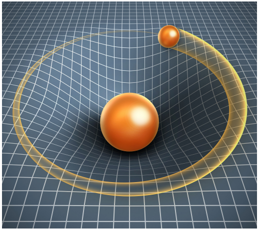
General relativity: gravity = curved space–time

- Massive bodies curve the spacetime!

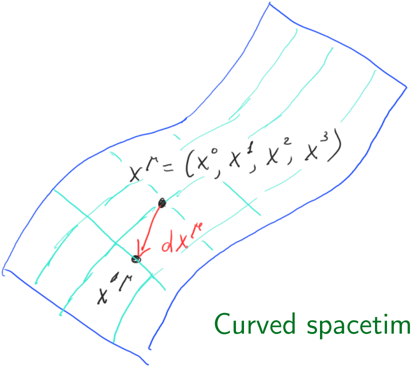
$$\underbrace{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R}_{\text{curvature}} = 8\pi G \underbrace{T_{\mu\nu}}_{\text{matter}}$$

- Flat spacetime: $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

$$\eta_{\mu\nu} = \mathrm{diag}(-1,\, 1,\, 1,\, 1)_{\mu\nu}$$



GR as a sofa



Curved spacetime

- Curved spacetime: $ds^2 = \underbrace{g_{\mu\nu}(x)}_{\text{metric}} dx^\mu dx^\nu$
  - Changes in causal structure: $ds^2 \lessgtr 0$
  - Curvature: $R_{\mu\nu} = R_{\mu\nu}[g]$

    $\rightarrow$ coordinate-covariant $x^\mu \rightarrow x'^\mu(x)$

    $\rightarrow$ characterizes geometry

    $\rightarrow$ explicit formula

Inset: coordinate invariance

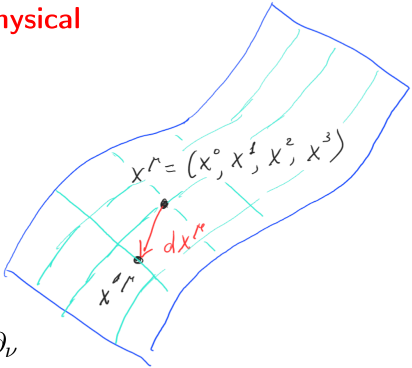
Main principle: coordinates are unphysical

- $x^\mu \to x'^\mu = x'^\mu(x)$ — coords
- $\underbrace{dx^\mu}_{\text{vector}} \to dx'^\mu = \underbrace{\dfrac{\partial x'^\mu}{\partial x^\nu}}_{\text{Jacobian}} \cdot dx^\nu$

$$A^\mu \to A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \cdot A^\nu$$

- $\underbrace{\partial_\mu}_{\text{covector}} \equiv \dfrac{\partial}{\partial x^\mu} \to \partial'_\mu = \underbrace{\dfrac{\partial x^\nu}{\partial x'^\mu}}_{\text{inverse J}} \cdot \partial_\nu$

$$T^\mu_\nu \to T'^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\lambda} \cdot \frac{\partial x^\rho}{\partial x'^\nu} \cdot T^\lambda_\rho \text{ — tensor}$$

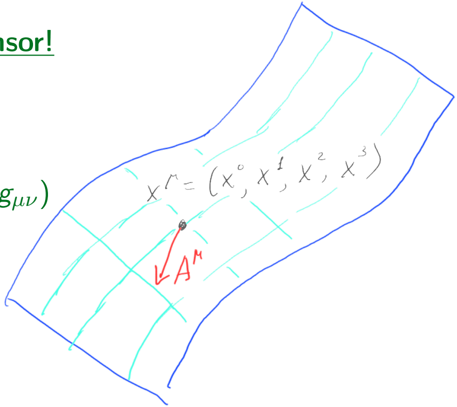


Why? $\left\{ \begin{array}{c} A^\mu B_\mu \\ A^\nu C^\mu g_{\mu\nu} \\ \ldots \end{array} \right\}$ — **invariants!**

Inset: curvature

- $\partial_\mu A^\nu$ — **not a tensor!**
- $\underbrace{\nabla_\mu}_{\text{covariant derivative}} A^\nu \equiv \partial_\mu A^\nu + \underbrace{\Gamma^\nu_{\mu\lambda}}_{\text{Christoffel symbols}} A^\lambda$ — **tensor!**
- $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)$
- Derivatives do not commute!

$$[\nabla_\mu \nabla_\nu]\, A^\lambda = - \underbrace{R_{\mu\nu\rho\cdot}{}^{\lambda}}_{\text{curvature tensor}} A^\rho$$



- $R^\mu{}_{\nu\lambda\rho} \equiv \partial_\lambda \Gamma^\mu_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\nu\lambda}$, $R_{\nu\rho} \equiv R^\mu{}_{\nu\mu\rho}$, $R \equiv g^{\mu\nu} R_{\mu\nu}$ nonzero in curved spacetime!
- **Now,** $T^\mu_\nu \nabla_\mu A^\nu$, $R_{\mu\nu} A^\mu B^\nu$ — **invariants!**

Example: Black hole spacetime

- Flat in radial coordinates:

$$ds^2 = -dt^2 + dr^2 + \underbrace{d\theta^2 + \sin^2\theta\, d\varphi^2}_{d\Omega^2}$$

- Black hole (Gullstrand–Painlevé coord-s):

$$ds^2 = -dt'^2 + (dr - v dt')^2 + r^2 d\Omega^2$$

- Spacetime velocity: $v(r) = -\sqrt{\dfrac{2GM}{r}}$
- M — black hole mass
- Schwarzshild radius = horizon:

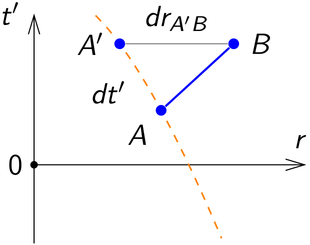
$$|v(r_h)| = 1 \text{ or } \boxed{r_h = 2GM}$$

→ $r < r_h$ — fall into the center!

→ $r = r_h$ — light stays in place

→ $r > r_h$ — fly away

Horizon is a fictituous surface!



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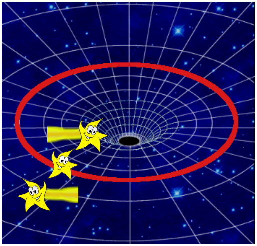

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Properties of black holes

- Black hole (Gullstrand–Painlevé coord-s):

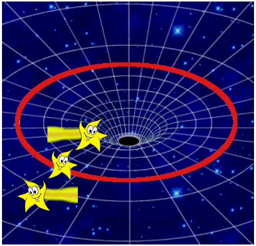
$$ds^2 = -dt'^2 + (dr - v dt')^2 + r^2 d\Omega^2$$

- Black hole (Schwarzshild coord-s):

$$t' \to t = t' + 2\sqrt{r r_h} + r_h \ln\left(\frac{\sqrt{r}-\sqrt{r_h}}{\sqrt{r}+\sqrt{r_h}}\right)$$

and obtain

$$ds^2 = -\underbrace{(1 - r_h/r)}_{f(r)} dt^2 + \frac{dr^2}{1 - r_h/r} + r^2 d\Omega^2$$



Properties

- GR solution: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$ without matter!
- Not singular at $r = r_h$
- But: singularity at $r \to 0$!

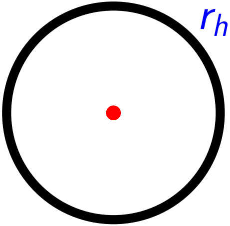
$$R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = 12 \frac{r_h^2}{r^6}$$

- Flat at $r \to +\infty$
  - $\to$ compact object
  - $\to$ Newtonian physics there!

$$U_N \approx \frac{1}{2}[f - 1] = -\frac{GM}{r}$$

Sizes of black holes

$$\boxed{r_h = 2GM}$$



- Astrophysical black holes (observed):

  $M = M_\odot \Rightarrow r_h \sim 3$ km

- Supermassive black holes (observed):

  $M = 10^6 \div 10^9\ M_\odot \Rightarrow r_h \sim 1 \div 1000\ R_\odot$

- Some other black holes (unobserved):

  $M \sim 3 \times 10^{-8}\ M_\odot \Rightarrow r_h \sim 0.1$ mm (Moon mass)

  $M \sim 10^{-12}\ M_\odot \Rightarrow r_h \sim 3$ nm (asteroid mass)

  $M \sim \text{kg} \Rightarrow r_h \sim 10^{11}\ \text{GeV}^{-1}$

  $M \sim M_{pl} \equiv G^{-1/2} \Rightarrow r_h \sim M_{pl}^{-1} \sim 10^{-33}\,\text{cm}$ (Planckian)

Black holes are small!

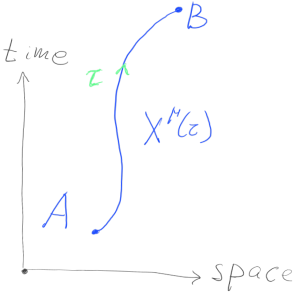
Moving in black hole spacetime

- Objects in GR move along geodesics
- Geodesics = maximum of proper time

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{-ds^2}$$

- Extremum ⇒ equation

$$\underbrace{\frac{du^\mu}{d\tau}}_{\text{acceleration}} + \underbrace{\Gamma^\mu_{\nu\lambda} u^\nu u^\lambda}_{\text{grav. force}} = 0$$



- Four-velocity: $u^\mu = \dfrac{dx^\mu}{d\tau}$, $\quad g_{\mu\nu} u^\mu u^\nu = -1$
- Light-like geodesics: same equation, but $g_{\mu\nu} u^\mu u^\nu = 0$
- Conserved quantities:
  - → Symmetry $x^\mu \to x^\mu + \xi^\mu \Delta t$, $\xi^\mu = (1, 0, 0, 0)$
  - → Energy $\boxed{E/m = -\xi^\mu u^\nu g_{\mu\nu}}$
  - → Angular momentum: $L/m = \eta^\mu u^\nu g_{\mu\nu}$, $\quad \eta^\mu = (0, 0, 0, \Delta\varphi = 1)$

Moving in black hole spacetime

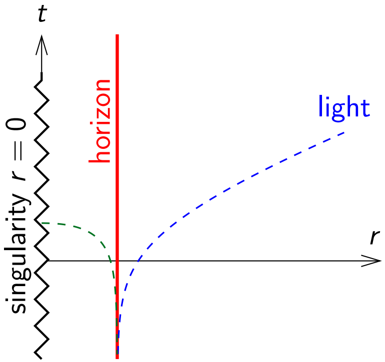
- Light-like geodesics
- Energy: $\boxed{\mathcal{E} = -\xi^\mu u^\nu g_{\mu\nu} = \pm \frac{dr}{d\tau}}$

  $\left(\text{because } u^2 = 0 \leftrightarrow f \frac{dt}{d\tau} = \pm \frac{dr}{d\tau}\right)$
- Moving in: $\mathcal{E} = -\frac{dr}{d\tau} > 0$
- Moving out: $\mathcal{E} = +\frac{dr}{d\tau}$

  $\rightarrow \mathcal{E} > 0$ outside horizon

  $\rightarrow \mathcal{E} < 0$ inside horizon

  Cannot leave the horizon!



out-moving light rays

Geodesics $\longleftrightarrow$ tests of GR

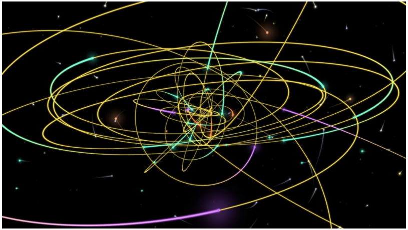
A portrait of a black hole

Supermassive black hole

galaxy: M87 distance: 5×10^7 ly observed on March 11, 2017 by Event Horizon Telescope	mass: $M \approx 6 \times 10^9\ M_\odot$ radius: $r_h \approx 100$ AU

Black hole in our galaxy

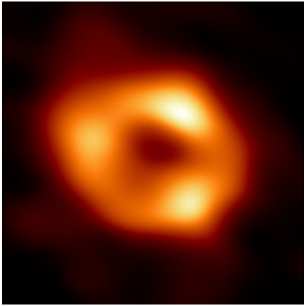
VLT



stars moving around black hole

Supermassive black hole
in Milky Way (Sgr A)

Mass: $M \sim 4 \cdot 10^6 M_\odot$
Radius: $r_h \sim 4\, R_\odot$
Distance: 8 kpc



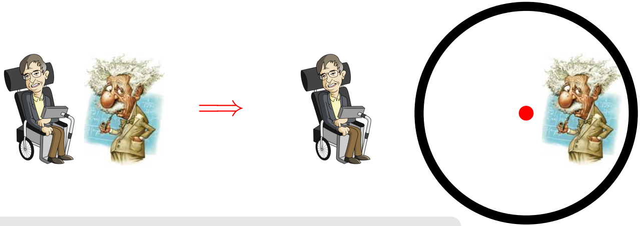
black hole portrait (EHT)

Is black hole singular?

Singular solutions do not exist in physics!

Argument

1. Give rockets to experimentalists
2. Let them go into black hole & measure the singularity
3. None of them returns!

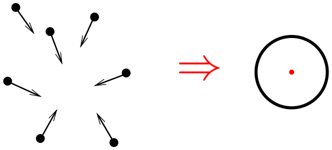


⇒ **Singularity is unobservable!**
... as well as the region inside the horizon

Singularity theorems: general idea

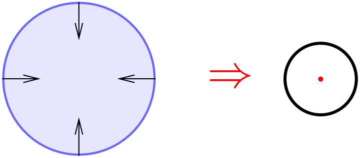
Singularities form from smooth matter distributions!

collapse of particles



particles black hole

collapse of a star



star black hole

supernovae SN2018gv →
(Hubble Space Telescope)

Inset: Energy–momentum tensor

- **Perfect fluid:**

$$T_{\mu\nu} = \underbrace{p}_{\text{pressure}} g_{\mu\nu} + (p + \underbrace{\rho}_{\text{density}})u^{\mu} \underbrace{u^{\nu}}_{\text{velocity}}$$

- Fluid at rest in flat space: $g_{\mu\nu} = \eta_{\mu\nu}$,

$$u^{\mu} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

- **Null Energy Condition (NEC):**

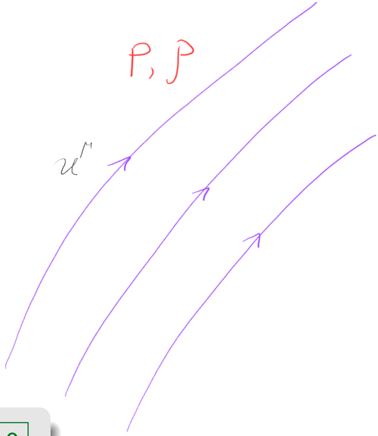
$T_{\mu\nu}k^{\mu}k_{\nu} \geq 0$ for any null k $\quad \boxed{k^{\mu}k^{\nu}g_{\mu\nu} = 0}$

$\rightarrow$ fluid: $p + \rho \geq 0$

$\rightarrow$ vacuum: $p + \rho = 0$

$\rightarrow$ **any sensible matter satisfies NEC!**

(the weakest condition on $T_{\mu\nu}$)



Singularity theorem

Roger Penrose '65

Trapped surface

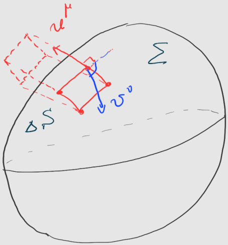
Two-dimensional surface Σ:

→ closed;

→ outer & inner light rays **converge**

$$u^\mu,\ v^\nu \perp \Sigma, \quad u^2 = v^2 = 0$$

$$\theta_u \equiv \underbrace{\nabla_u \Delta S}_{-K=-\nabla_\mu u^\mu} \leq 0, \quad \theta_v \leq 0$$



theorem

Suppose:

- GR equations are valid;
- matter satisfies NEC;
- trapped surface exists

Then:

- singularity forms;
- trapped surface collapses (gravity is attractive ↔ rays converge)

Singularities form from smooth initial data!

hence black holes exist...

Cosmic censorship hypothesis

Roger Penrose '69

Hypothesis

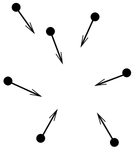
→ Singularities in GR are covered by horizons

→ **If the matter is good**

dust: not good

perfect fluid: not good

needs pressure & friction!



Nobel prize to R. Penrose

"for the discovery that black hole formation is a robust prediction of the general theory of relativity"

for the conjecture!

By default, believe in conjecture!

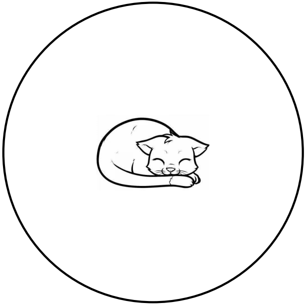
Simplification: hoop conjecture

Kip Thorne '72

Body of mass M collapses if:

- it fits into a hoop of radius $r_h = 2GM$;
- irrespectively of the hoop orientation

Squeeze them:



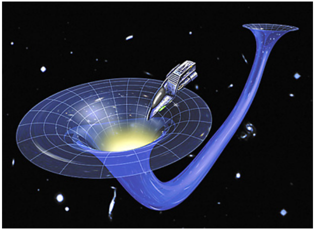
Example 1: Sun, $M = M_\odot$, $R \sim 10^6$ km $\Rightarrow$ $\boxed{r_h = 3 \text{ km}}$

Example 2: Neutron star, $M \sim 1.5\, M_\odot$, $R \sim 10$ km $\Rightarrow$ $\boxed{r_h = 4 \text{ km}}$

Can you avoid singularity?

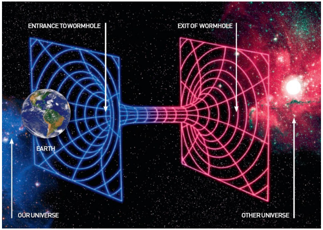
Collapse into a wormhole?

our Universe



throat $\leftrightarrow$ throat

between universes



throat $\leftrightarrow$ throat

- Move throat with acceleration
  $\Rightarrow$ paradox of twins
  $\Rightarrow$ Time Machine!

- Need ~~NEC~~ for that!
  ghosts,
  beyond Hordenski theories,
  etc...

The fact is: singularity is unobservable!

Penrose diagrams

Where can you go? $\leftarrow$ causal structure!

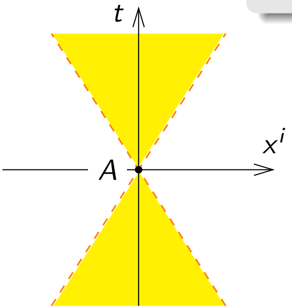
Draw a diagram:

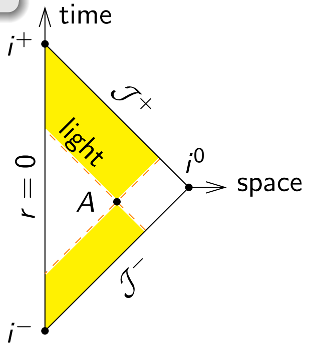
- $\rightarrow$ ignore θ and φ;
- $\rightarrow$ keep time & radius;
- $\rightarrow$ light rays are diagonal (45°)!
- $\rightarrow$ draw infinity as a finite box.

Notations

$i^{\pm}$ — past/future infinities
i^{0} — space infinity
$\mathcal{J}^{\pm}$ — null infinities

flat spacetime





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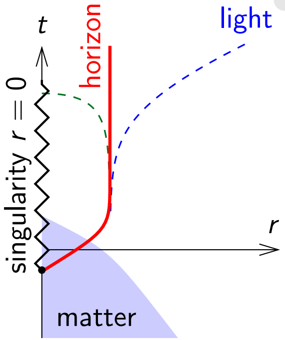
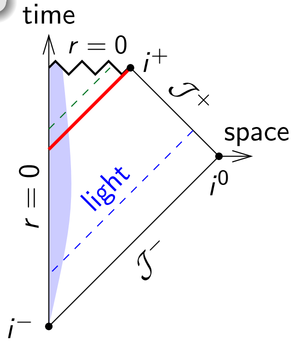
matter collapse





Definition of a black hole

- Black hole = spacetime region
  from where you cannot go to infinity

  Nonlocal: depends on the future!

- Horizon = black hole boundary

  → also nonlocal in time

  → fictitious surface

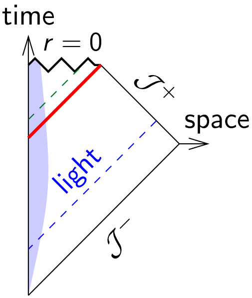
  → may have small curvature

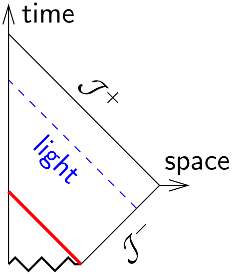
- White hole = spacetime region
  where you cannot enter

  → time reflection $t \to -t$

  → singular spacetime — does not exist?

Black holes are defined by causal structure!





Summary

- **GR:**
  - gravity = curved spacetime or $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$
  - curvature: $R = R[g]$
  - energy-momentum tensor: $T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) u_\mu u_\nu$ (fluid)
  - bodies move along geodesics (extremal proper time)
    $\rightarrow$ symmetries $\leftrightarrow$ conserved quantities
- **Classical black holes**
  - Black hole = a region you cannot leave (N.B. moving liquid)
    $\rightarrow$ $E < 0$ inside black hole (out-moving rays)
  - horizon = black hole boundary, $r_h = 2GM$
    $\rightarrow$ fictitious surface (nonlocal in time)
  - singularities form in collapse of smooth matter
    $\rightarrow$ if trapped surface appears (theorem)
    $\rightarrow$ if hoop conjecture is satisfied
    $\rightarrow$ censorship conjecture: sing-s are covered by horizons!
  - BHs are smooth: $r = 0$ singularity is unobservable
    $\rightarrow$ see causal structure in Penrose diagram
  - Classical BHs are the graveyards of the Universe!

Thank you for attention!